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Double Complexes and Cohomological Hierarchy in a Space of Weakly Invariant Lagrangians of Mechanics

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For a given configuration space M and Lie algebra $\mathcal{G}$ whose action is defined on M the space $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians (i.e. Lagrangians whose motion equations left hand sides are $\mathcal{G}$-invariant) is studied. The problem is reformulated in terms of the double complex of Lie algebra cochains with values in the complex of Lagrangians. Calculating the cohomology of this complex using the method of spectral sequences we arrive at the hierarchy in the space $\mathcal{V}_{0.0}$: The double filtration $\{\mathcal{V}_{s.\sigma}\}$ $(s = 0, 1, 2, 3, 4, \sigma = 0, 1)$ and the homomorphisms on every space $\mathcal{V}_{s.\sigma}$ are constructed. These homomorphisms take values in the cohomologies of the algebra $\mathcal{G}$ and configuration space M. On one hand every space $\mathcal{V}_{s,\sigma}$ is the kernel of the corresponding homomorphism, on the other hand this space is defined by its physical properties.

I Introduction

The cohomology of the symmetries algebra has important consequences for properties of corresponding theory [1,2] and cohomological methods play essential role in many problems of modern fields theory. For example their application made more clear the understanding of algebraic origin of gauge anomalies. As it was shown in [1] one can consider axial anomalies of four-dimensional gauge theory in terms of infinitesimal cocycles in a representation of gauge group.

Another example is BRST formalism which at beginning was formulated in terms of symplectic geometry of phase space expanded by the ghosts and antighosts, then it was understood [3,4,5,6,7] that the language of homological algebra is more deeply related with physical meaning of this formalism: Inclusion of ghosts and antighosts corresponds to the construction of the chain of free modules (free resolvent) on phase space of constrained system where the constrains cannot be resolved in a direct way. The operator corresponding to BRST charge becomes the differential of the complex of these resolvents. Further the investigation of local BRST cohomology was performed with use of developed homological methods. (See [8,9,10] with citations there.)

In this paper we consider more modest problem. We study relations between Noether identities and related phenomena for global symmetries of Lagrangians and cohomological properties of the algebra of these symmetries.

Our considerations will be carried out for mechanics but the scheme has straightforward generalization on the case of field theory Lagrangians.

The standard statement (Noether 1-st Theorem) is that if the Lagrangian L is invariant under the action of Lie algebra $\mathcal{G}$ of rigid symmetries $\{\delta_i\}$ then to every symmetry δ_i corresponds the charge $N_i(L)$ which is preserved on the equations of motion [11].

If to $\{\delta_i\}$ corresponds the Lie algebra of vector fields $\{\mathbf{X}_i = X_i^\mu \frac{\partial}{\partial q^\mu}\}$ (infinitesimal transformations of configuration space) then $\delta_i q^\mu \sim X_i^\mu$,

$$\delta_i L = 0 = \mathcal{L}_{\mathbf{X}_i} L = X_i^\mu \mathcal{F}_\mu(L) + \frac{d}{dt}\left(X_i{}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha}\right) \text{ and}$$

$$N_i = X_i{}^\mu \frac{\partial L}{\partial \dot{q}^\mu}, \qquad \text{where} \qquad \mathcal{F}_\mu(L) = \frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu} \tag{1.1}$$

is the left hand side (l.h.s.) of the equations of motion $\mathcal{F}_\mu = 0$ of the Lagrangian L.

The statement of Noether theorem is valid also in a case if under the actions of transformations $\{\delta_i\}$ Lagrangian is preserved up to a full derivative of some functions $\{\alpha_i(q)\}$

$$\delta_i L = 0 \to \delta_i L = d\alpha_i, \text{ then } N_i(L) \to N_i(L) - \alpha_i\,. \tag{1.2}$$

At what extent this full derivative is essential? The redefinition of L on a full derivative $L \to L + df$ changes α_i on $\alpha_i + \delta_i f$. The algebra of symmetries of the Lagrangian can be considered as generalized if $d\alpha_i$ is not equal 0 in (1.2), and it is *essentially generalized* if it cannot be canceled by redefinition of Lagrangian on a full derivative i.e. $\delta_i L = d\alpha_i$ but the equations

$$d(\alpha_i + \delta_i f) = 0\,. \tag{1.3}$$

have no solutions.

Using the basic properties of operators δ and d: $\delta^2 = d^2 = 0$, $d\delta = \delta d$ (see the Section II) we obtain from (1.2) that

$$0 = \delta^2 L = \delta d\alpha_i = d\delta\alpha_i, \text{ so } (\delta\alpha)_{ij} = w_{ij} = constant, \tag{1.4}$$

where $(\delta\alpha)_{ij} = \mathcal{L}_i\alpha_j - \mathcal{L}_j\alpha_i - \alpha_k c_{ij}^k$ and c_{ij}^k are structure constants of the symmetries Lie algebra.

It is easy to see that w_{ij} is the cocycle of algebra $\mathcal{G}$ in constants. In a case if w_{ij} is not coboundary one can see that the symmetries are essentially generalized. Indeed if according to (1.3) $\alpha_i = -\delta_i f + t_i$ where t_i are constants then w_{ij} in (1.4) is coboundary in constants: $w_{ij} = (\delta t)_{ij} = -c_{ij}^k t_k$.

Let us consider for example the algebra of space translations. This algebra has 2-cohomology in constants which are represented by antisymmetric tensors B_{ik}. (This algebra is abelian, so $\delta B = 0$ and the equation $B = \delta t$ has no solutions in constants.) To obtain Lagrangian which possesses generalized translation symmetries corresponding to these cocycles, we note that for this Lagrangian $\alpha_i = A_{ij} q^j$. By redefinition of a Lagrangian on a full derivative one can reduce A_{ij} to antisymmetric tensor and we come to Lagrangian

$$L = f(\dot{q}) + q^i B_{ij} \dot{q}^j\,. \tag{1.5}$$

If $f(\dot{q}) = \frac{m\dot{q}^2}{2}$ it is the well-known Lagrangian of particle in constant magnetic field.

In the Section V we consider an analogous statement for Galilean group: we show that one comes to the Lagrangian of free particle as a unique Lagrangian corresponding to Bargman cocycle of Lie algebra of Galilean group.

We see that one of the reasons of generalized symmetries appearing is the existence of 2-cohomology of corresponding Lie algebra [2,12]. Of course situation is more complicated. For example by evident reasons for this phenomenon is responsible de Rham cohomology of configuration space. If L_{inv} is $\mathcal{G}$-invariant Lagrangian and $L = L_{inv} + A_\mu(q)\dot{q}^\mu$ where $A_\mu(q)dq^\mu$ is a closed differential 1-form which is not exact $(A_\mu(q)dq^\mu \neq df)$ then in general L is not $\mathcal{G}$-invariant. It has the same equations of motion but it differs from L_{inv} on Aharonov-Bohm like effects.

Even in the case if de Rham cohomology is trivial and the cocycle w_{ij} in (1.4) is coboundary the symmetries of Lagrangian can be essentially generalized. The coboundary condition $w_{ij} = -c_{ij}^k t_k$ is necessary but not sufficient for (1.3) to have a solution. It is another cohomologies of symmetries algebra which prevent a Lagrangian to be reduced to $\mathcal{G}$-invariant by redefinition on a full derivative.

The purpose of our paper is to investigate systematically this phenomenon.

For the algebra $\mathcal{G}$ of vector fields on the configuration space M and a Lagrangian $L(q,\dot{q})$ on M we considered the following possible cases of generalized symmetries appearing

1) The action of $\mathcal{G}$ on the Lagrangian L produces the 2-cocycle on $\mathcal{G}$:
$$\delta_i L(q,\dot{q}) = \frac{d}{dt}\alpha_i(q),\ w_{ij} = \delta_i\alpha_j - \delta_j\alpha_i - c_{ij}^k\alpha_k\,.$$

2) The action of $\mathcal{G}$ on the Lagrangian L produces the 2-cocycle, but it is trivial:
$$w_{ij} = -c_{ij}^k t_k\,.$$

3) The Lagrangian L differs from invariant one on a closed form:
$$L = L_{inv} + A_\mu(q)\dot{q}^\mu, (\partial_\mu A_\nu - \partial_\nu A_\mu = 0)$$
hence $\delta_i L = \frac{d}{dt}(A_\mu X_i^\mu)$ and $w_{ij} = 0$.

4) The Lagrangian L differs from $\mathcal{G}$-invariant one on an exact form (full derivative):
$$L = L_{inv} + \partial_\mu f(q)\dot{q}^\mu = L_{inv} + \frac{d}{dt}f(q),\ \ \delta_i L_{inv} = 0. \tag{1.6}$$

One can see that
$$\text{"4"} \Rightarrow \text{"3"} \Rightarrow \text{"2"} \Rightarrow \text{"1"} \tag{1.7}$$

We briefly discuss how the generalized symmetries reveal itself in Hamiltonian mechanics and in a quasiclassical approximation of quantum mechanics.

If the Lagrangian is $\mathcal{G}$-invariant, then to the Noether charges $N_i(L)$ in (1.1) in Hamiltonian framework correspond the charges $N_i^{ham} = X_i^\mu p_\mu$. They generate the $\mathcal{G}$-algebra structure via Poisson brackets
$$\{N_i^{ham}, N_j^{ham}\} = c_{ij}^k N_k^{ham}\,. \tag{1.8}$$

In quasiclassical approximation of quantum mechanics to these charges correspond the operators $X_i^\mu \hat{p}_\mu$. Their action on quasiclassical wave function in configuration representation is reduced to infinitesimal transformation of wave functions argument:
$$\imath\hat{\delta}_i\Psi = \Psi(q^\mu + \delta_i q^\mu) - \Psi(q^\mu).$$
In the case if symmetries algebra is generalized, one can see that correspondingly to (1.2)

$$N_i^{ham} = X_i^\mu p_\mu - \alpha_i .$$

The corresponding operators act not only on quasiclassical wave functions argument but on its phase too:

$$\hat{\delta}_i \Psi = -\imath X_i^\mu \frac{\partial \Psi(q)}{\partial q^\mu} + \imath \alpha_i(q) \Psi(q^\mu) \,. \tag{1.9}$$

In the case if the Lagrangian does not possess the property "2" in (1.6), i.e. the generalized symmetries lead to non-trivial cocycle, the Lie algebra of Hamiltonian Noether charges N_i^{ham} is the central extension of the Lie algebra $\mathcal{G}$ which corresponds to the cohomology class of the cocycle w_{ij}.

$$\{N_i^{ham}, N_j^{ham}\} = c_{ij}^k N_k^{ham} + w_{ij} \,. \tag{1.10}$$

Correspondingly in this case in (1.9) is realized an essentially projective representation of the Lie algebra $\mathcal{G}$.

In the case if the Lagrangian possesses the property "2" in (1.6), then one can choose α_i such that (1.8) is satisfied and the quantum representation (1.9) of $\mathcal{G}$ becomes linear. But if this Lagrangian does not possess the property "4" in (1.6) then the action of quantum transformation on the phase factor cannot be removed by redefinition $\Psi \to e^{if}\Psi$ of the wave function corresponding to redefinition of Lagrangian on a full derivative. In this case one can say that the linear transformation (1.9) is splitted on a space-like transformation+intrinsic spin-like transformation. Nevertheless if the Lagrangian possesses the property "3", i.e. it differs of an invariant Lagrangian on Bohm-Aharonov like effects, then the action on phase in (1.9) can be removed locally.

We call time-independent Lagrangian $L(q,\dot{q})$ *weakly $\mathcal{G}$-invariant* if l.h.s. of its motion equations (1.1) is $\mathcal{G}$-invariant. For example the Lagrangian L in (1.2) is weakly $\mathcal{G}$-invariant. One can show that if L is weakly $\mathcal{G}$-invariant Lagrangian then

$$\delta_i L = c_i + w_i , \tag{1.11}$$

where c_i are constants and w_i correspond to closed forms: $w_i = w_{i\mu}(q)\dot{q}$ where differential 1-forms $w_{i\mu}(q)dq^\mu$ are closed. (See in details later.)

If $\{w_i\}$ correspond to exact forms: $w_{i\mu}(q)dq^\mu = da_i(q)$, $w_{i\mu}(q)\dot{q}^\mu = \partial_\mu \alpha_i(q)\dot{q}^\mu = d\alpha_i(q)/dt$ and

$$c_i = 0 \tag{1.12}$$

then we come to (1.2). In the case if (1.12) is not obeyed the corresponding Noether charges

$$N_i = X_i{}^\mu \frac{\partial L}{\partial \dot{q}^\mu} - \alpha_i - c_i t \tag{1.13}$$

depend on time.

We denote by $\mathcal{V}_{0.0}$ the space of weakly $\mathcal{G}$-invariant Lagrangians on M and by $\mathcal{V}_{0.1}$ the subspace of $\mathcal{V}_{0.0}$ for which the condition (1.12) is satisfied. We denote by $\mathcal{V}_{s.1}$ ($s = 1,2,3,4,$) the space of Lagrangians for which the property "s" in (1.6) is satisfied. According to (1.7)

$$\mathcal{V}_{4.1} \subseteq \mathcal{V}_{3.1} \subseteq \mathcal{V}_{2.1} \subseteq \mathcal{V}_{1.1} \subseteq \mathcal{V}_{0.1} \subseteq \mathcal{V}_{0.0} \,. \tag{1.14}$$

One can consider also subspaces $\{\mathcal{V}_{s.0}\}$ of the space $\mathcal{V}_{0.0}$

$$\mathcal{V}_{4.0} \subseteq \mathcal{V}_{3.0} \subseteq \mathcal{V}_{2.0} \subseteq \mathcal{V}_{1.0} \subseteq \mathcal{V}_{0.0}, \quad \mathcal{V}_{s.1} \subseteq \mathcal{V}_{s.0} \tag{1.15}$$

which correspond to $\{\mathcal{V}_{s.1}\}$ if we ignore the condition (1.12): The weakly $\mathcal{G}$-invariant Lagrangian L belongs to $\mathcal{V}_{1.0}$ if $\delta_i L = d\alpha_i + c_i$. It is easy to see that in this case $\delta\alpha$ is also 2-cocycle as in (1.4). $L \in \mathcal{V}_{2.0}$ if this cocycle is trivial, $L \in \mathcal{V}_{4.0}$ if $\alpha_i = \delta_i f$ and $L \in \mathcal{V}_{3.0}$ if it differs from $\mathcal{V}_{4.0}$ on a closed form. Lagrangians in $\mathcal{V}_{s.0}$ in general have time-dependent Noether currents (1.13).

What can we say more about embeddings (1.14, 1.15)? Does weakly $\mathcal{G}$-invariant Lagrangian possesses generalized symmetries (1.2)? Can it be reduced to $\mathcal{G}$-invariant? Does there exist Lagrangian which belongs to the space $\mathcal{V}_{s.0}$ and which does not belong to the space $\mathcal{V}_{s+1.0}$ or $\mathcal{V}_{s.1}$? If an answer is "no" what are the reasons of it.

To answer on these questions we establish the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians. This hierarchy relates the phenomena discussed above with cohomologies groups of the Lie algebra $\mathcal{G}$ and the configuration space M.

The scheme of our considerations is the following. We are fixing configuration space M and finite-dimensional algebra $\mathcal{G}$ of its transformations. Then we establish relations between weakly $\mathcal{G}$-invariant Lagrangians on M and the cohomologies of algebra $\mathcal{G}$ and M. From considerations above we see that in the phenomena which we are investigating are interplaying two differentials δ and $d_{_{E.L}}$ where the differential δ corresponds to the symmetries and $d_{_{E.L}}$ is the prolongation of exterior differential which acts on Lagrangians. (It is variational derivative, whose action leads to Euler-Lagrange equation. See in details the Section II). These differentials as well as differentials d and δ satisfy the conditions: $\delta^2 = d_{_{E.L}}^2 = d_{_{E.L}}\delta - \delta d_{_{E.L}} = 0$. We naturally come to the differential $Q = d_{_{E.L}} \pm \delta$ which is strictly related with our problem. For example to condition $\delta L = d\alpha$ in (1.2) corresponds the condition $Q(L, \alpha_i) = (d_{_{E.L}} L, 0, w = \delta\alpha)$. To redefinition of Lagrangian on a full derivative $L \to L + df$ corresponds the changing of the cochain (L, α_i) on coboundary: $(L, \alpha_i) \to (L, \alpha_i) + Qf = (L + df, \alpha_i + \delta_i f)$.

It is the cohomology of the differential Q which allow us to reveal the relations between generalized symmetries of Lagrangians and cohomologies of the configuration space and the symmetries Lie algebra. We do it in a following way. Using a technique of spectral sequences we calculate the cohomology of Q via cohomology of $d_{_{E.L}}$ by modulo δ, then vice versa via cohomology of δ by modulo $d_{_{E.L}}$. Calculating cohomology of operator Q in the first way we come to the spaces K_s which are expressed in terms of cohomologies of Lie algebra and configuration space. On the other hand, calculating the same cohomology in the second way, we come naturally to the space $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians and to its subspaces $\{\mathcal{V}_{s.\sigma}\}$ (1.14,1.15). Natural relations which arise between the results of calculations in the first and in the second way lead to the sequence of homomorphisms between the spaces $\{\mathcal{V}_{s.\sigma}\}$ and $\{K_s\}$ which define these spaces in a recurrent way via the kernels of corresponding homomorphisms.

This construction establishes hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians making links between the physical properties of Lagrangians and pure mathematical objects: The condition that Lagrangian belongs to some space $\mathcal{V}_{s.\sigma}$ and it *does not belong to the space* $\mathcal{V}_{s+1.\sigma}$ or $\mathcal{V}_{s.\sigma+1}$ in terms of this hierarchy is reformulated to the condition that the value of the corresponding homomorphism on it, is not equal to zero. The problem of analyzing the content of the spaces $\{\mathcal{V}_{s.\sigma}\}$ and their differences is reduced to the problem of calculating the corresponding homomorphisms. For example in the case if the space K_s

is trivial, then $\mathcal{V}_{s-1.\sigma} = \mathcal{V}_{s.\sigma}$. In particular if all the spaces K_s are trivial then all weakly invariant Lagrangians are invariant (up to a full derivative).

The plan of the paper is the following.

In the Section II we consider the complex of Lagrangians, clarify its relations with corresponding complex of differential forms.

In the Section III we present the calculations of cohomology of the differential Q of the double complex of cochains which are defined on the Lie algebra $\mathcal{G}$ and take values in the functions on M and in Lagrangians of classical mechanics. Using the results of these calculations in the Section IV we establish hierarchy in the space of weakly invariant Lagrangians and consider some general properties of this hierarchy. It is the main result of the paper. In this Section from our point of view we consider also the hierarchy for Lagrangians polynomial in velocities.

In the Section V using this hierarchy we calculate the content of the subspaces $\mathcal{V}_{s.\sigma}$ in (1.14, 1.15) for some special cases of configuration spaces and symmetries algebras. In particular we perform this analysis for $so(3)$, Poincaré and Galilean algebras.

In the Section VI we give some motivations for the technique we used in this paper.

In Appendixes we give a brief sketch on the notion of Lie algebra cohomology and calculation of double complexes cohomology via corresponding spectral sequences.

II The complexes of Lagrangians and Differential Forms

Let M be an n-dimensional manifold (configuration space) and $\mathcal{G}$ be Lie algebra acting on it. It means that it is defined the homomorphism Φ from $\mathcal{G}$ in the Lie algebra of vector fields on M:

$$\mathcal{G} \ni x \stackrel{\Phi}{\longrightarrow} \tilde{x} \text{ (fundamental vector fields)} : \quad [\tilde{x}, \tilde{y}] = \widetilde{[x, y]} \,. \tag{2.1}$$

We denote this construction by $[\mathcal{G}, M]$ pair.

Let $\Omega^q(M)$ be the space of differential q-forms on M. The linear spaces $\Omega^q(M)$ for any given q can be considered as modules on $\mathcal{G}$ if we define the action of algebra on forms via Lie derivatives along corresponding fundamental vector fields: $h \circ w = \mathcal{L}_{\tilde{h}} w$. One can consider the $\mathcal{G}$-differential corresponding to this module structure and cohomologies spaces $H^q(\mathcal{G}, \Omega^q(M))$ which are $\mathcal{G}$-cohomologies with coefficients in $\Omega^q(M)$. (See Appendix 1).

On the manifold M endowed with the action of Lie algebra $\mathcal{G}$ one can also consider usual de Rham cohomologies $H^q(M)$ of the differential forms complex $\{\Omega^q, d\}$, where d is exterior differential. One can naturally prolongate the action of exterior differential d from the spaces $\Omega^q(M)$ (0-cochains) on the spaces $C^p(\mathcal{G}, \Omega^q(M)) = C^p(\mathcal{G}) \otimes \Omega^q(M)$ of p-cochains on the Lie algebra $\mathcal{G}$ with values in Ω^q, taking values of d on cochains in constants to be zero. The differentials d and δ commute with each other: $d\delta = \delta d$ and one can consider the corresponding double complex $\{C^p(\mathcal{G}, \Omega^q(M)), d, \delta\}$.

To include Lagrangians in a game we enlarge the complex $\{\Omega^q, d\}$ of differential forms to the complex $\{\Lambda^q(M), d_{_{E.L}}\}$ of Lagrangians, following [13].

We define the space $\Lambda^q(M)$ of q-Lagrangians ($q \geq 1$) as the space of functions (Lagrangians) which depend on points q^μ of manifold M and on derivatives $\frac{\partial q^\mu}{\partial \xi^\alpha}, \ldots, \frac{\partial^k q^\mu}{\partial \xi^{\alpha^1}, \ldots, \partial \xi^{\alpha^k}}$ of an arbitrary but finite order k of parameters $(\xi^1, \ldots, \xi^q)$ which take values in q-dimensional space $\mathbf{R}^q$. In the case $q = 0$ we put $\Lambda^0(M) = \Omega^0(M)$ is the

space of functions on M. We say that Lagrangian has the rank k if the highest degree of derivatives on whose it depends is equal to k and we denote by Λ^q_k the subspace of Λ^q which contains q-Lagrangians of the rank k. The Lagrangians of classical mechanics which we consider in the following Sections belong to Λ^1_1.

If L is the Lagrangian in $\Lambda^q(M)$ then to every map (q-dimensional path)

$$q^\mu(\xi^1,\ldots,\xi^q)\colon R^q\to M \tag{2.2}$$

corresponds the integral

$$S_L([q(\xi)])=\int L\left(q^\mu(\xi),\frac{\partial q^\mu(\xi)}{\partial\xi^\alpha},\ldots,\frac{\partial q^\mu(\xi)}{\partial\xi^{\alpha^1},\ldots,\partial\xi^{\alpha^k}}\right)d\xi^1\ldots d\xi^q\,. \tag{2.3}$$

This defines the natural embedding of the space $\Omega^q(M)$ of differential q-forms in $\Lambda^q_1(M)$:

$$w=w_{i_1\ldots i_q}(q)dq^{i_1}\wedge\ldots\wedge dq^{i_q}\longrightarrow L_w=n!w_{i_1\ldots i_q}(q)\frac{\partial q^{i_1}}{\partial\xi^1}\cdot\ldots\cdot\frac{\partial q^{i_q}}{\partial\xi^q}\,. \tag{2.4}$$

The integral $S_{L_w}([q(\xi)])$ is equal to the integral of differential form w over the surface which is the image of the map (2.2). It does not depend on choice of parametrization $q(\xi)$ of this surface. We say that Lagrangian L_w corresponds to the differential form w and later on we often will not differ w and L_w.

Remark In general for an arbitrary Lagrangian the l.h.s. of (2.3) is not correctly defined on images of maps (2.2). It can be considered as functional on embedded surfaces which does not depend on its parametrization in a case if Lagrangian L is a *density*, i.e. under reparametrization $q(\xi)\to q(\xi(\tilde\xi))$, $L\to L\cdot det(\partial\xi/\partial\tilde\xi)$ (see for example [14,15.]). The Lagrangians corresponding to differential forms are the special examples of densities.

To define the complex of Lagrangians which is the generalization of de Rham complex we consider following [13] the differential $d_{_{E.L}}$, using Euler-Lagrange equations of motion for the functional (2.3):

$$d_{_{E.L}}\colon\Lambda^q\to\Lambda^{q+1},\quad d_{_{E.L}}L\left(q,\frac{\partial q^\mu}{\partial\xi^{\tilde a}},\ldots,\frac{\partial q^\mu}{\partial\xi^{\tilde a^1},\ldots,\partial\xi^{\tilde\alpha^k}}\right)=\mathcal{F}_\mu(L)\frac{\partial q^\mu}{\partial\xi^{q+1}}\,. \tag{2.5}$$

where $\tilde\alpha=(1,\ldots,q,q+1),\alpha=(1,\ldots,q)$ and $\mathcal{F}_\mu(L)$ are l.h.s. of Euler–Lagrange equations of the Lagrangian L, i.e. variational derivatives of the corresponding functional (2.3): $\mathcal{F}_\mu(L)=\frac{\delta}{\delta q^a}S_L([q(\xi^\alpha)])$

For example if $L\in\Lambda^q_1(M)$, $L=L(q,\frac{\partial q^\mu}{\partial\xi^a})$ then

$$d_{_{E.L}}L\left(q,\frac{\partial q^\mu}{\partial\xi^{\tilde a}},\frac{\partial^2 q^\mu}{\partial\xi^{\tilde a}\partial\xi^{\tilde\beta}}\right)=\left(\frac{\partial L}{\partial q^\mu}-\frac{\partial^2 L}{\partial q^\nu\partial q^\mu_\alpha}\frac{\partial q^\nu}{\partial\xi^\alpha}-\frac{\partial^2 L}{\partial q^\nu_\beta\partial q^\mu_\alpha}\frac{\partial^2 q^\nu}{\partial\xi^\alpha\partial\xi^\beta}\right)\frac{\partial q^\mu}{\partial\xi^{m+1}}\,. \tag{2.6}$$

(In general $d_{_{E.L}}\Lambda^q_k\subseteq\Lambda^{q+1}_{2k}$)

One can show that as well as for exterior differential d, $d^2_{_{E.L}}=0$ [13] and consider cohomology of the complex

$$\{\Lambda^q(M),d_{_{E.L}}\}\colon\quad\Lambda^0(M)\xrightarrow{d_{E.L}}\Lambda^1(M)\xrightarrow{d_{E.L}}\Lambda^2(M)\xrightarrow{d_{E.L}}\ldots\,. \tag{2.7}$$

From the definition of $d_{E.L}$ and (2.4) it follows that: $L_{dw} = d_{E.L} L_w$. The complex $\{\Omega^m(M), d\}$ of differential forms is subcomplex of the complex (2.7).

The spaces $\Lambda^q(M)$ of Lagrangians for any given q (and their subspaces $\Lambda^q_k(M)$ for any given q and k) as well as $\Omega^q(M)$ can be naturally considered as modules on Lie algebra $\mathcal{G}$ if we define the action of Lie algebra elements on Lagrangians via Lie derivative: if $x \in \mathcal{G}$ and $\tilde{x} = \Phi x = X^\mu(q)\partial/\partial q^\mu$ then

$$(x \circ L) = \mathcal{L}_{\tilde{x}} L = X^\mu \frac{\partial L}{\partial q^\mu} + D_\alpha X^\mu \frac{\partial L}{\partial q^\mu_\alpha} + D_\beta D_\alpha X^\mu \frac{\partial L}{\partial q^\mu_{\alpha\beta}} + \dots \tag{2.8}$$

where $D_\alpha = \frac{d}{d\xi^\alpha} = q^\mu_\alpha \frac{\partial}{q^\mu} + q^\mu_{\alpha\beta} \frac{\partial}{q^\mu_\beta} + \dots$ is the full derivative. If a Lagrangian corresponds to differential form then (2.8) corresponds to usual Lie derivative: $\mathcal{L} L_w = L_{\mathcal{L} w}$. To the identity $\mathcal{L}_\eta w = dw\rfloor\eta + d(w\rfloor\eta)$ for Lie derivative on forms corresponds the identity $\mathcal{L}_\eta L = \eta^\mu \mathcal{F}_\mu(L) + D_\alpha N^\alpha$ which leads to Noether currents N^α in the case if $\mathcal{L}_\eta L = 0$.

Considering $\mathcal{G}$-differential δ corresponding to this module structure we come to the spaces $H^p(\mathcal{G}, \Lambda^q_k(M))$ of $\mathcal{G}$-cohomologies with coefficients in $\Lambda^q_k(M)$.

In the same way like for differential forms one can prolongate the action of $d_{E.L}$ on the spaces $C^p(\mathcal{G}, \Lambda^q)$ of p-cochains with values in Λ^q and consider the double complex $\{C^p(\mathcal{G}, \Lambda^q), d_{E.L}, \delta\}$ because for Lagrangians $d_{E.L}$ and δ commute also. The complex $\{C^p(\mathcal{G}, \Omega^q), d, \delta\}$ is embedded in this complex.

The cohomology of the complex (2.7) evidently is different from de Rham cohomology, but on the other hand

Proposition 1 *

1.If Lagrangian L is exact: $L = d_{E.L} L'$ and it is a density (see the Remark above), then it corresponds to an exact differential form.

2. If Lagrangian L is closed and it depends only on first derivatives: $d_{E.L} L = 0, L \in \Lambda^q_1$, then it corresponds to closed differential form up to a constant

$$L = L_w + c,\ dw = 0\,. \tag{2.9}$$

In the case if L in (2.9) is a density then $c = 0$.

The 2-nd statement immediately follows from (2.6) and the definition of the density. The 1-st one we do not need here and we omit its proof.

We use this Proposition to consider the following subcomplex $(\mathcal{C}^*, d_{E.L})$ of the complex (2.7), which will be of use in this paper:

$$(\mathcal{C}^*, d_{E.L}): \qquad \Lambda^0(M) \xrightarrow{d_{E.L}} \Lambda^1_1(M) \xrightarrow{d_{E.L}} d_{E.L}\Lambda^1_1(M) \longrightarrow 0 \tag{2.10}$$

where as well as in (2.7) $\mathcal{C}^0 = \Lambda^0(M)$ is the space of functions on M; $\mathcal{C}^1 = \Lambda^1_1(M)$ is the space of Lagrangians $L(q^\mu, \dot{q}^\mu)$ of classical mechanics defined on the configuration space M,

---

* The complex (2.7) differs from the standard variational complex (See for example [16].) It was introduced by Th. Voronov in [13] for the Lagrangians on superspace. This complex and Proposition are useful in supermathematics where the concept of usual differential form is ill-defined [15,17].

$\mathcal{C}^2$ is the subspace of coboundaries in Λ_2^2. It contains elements corresponding to equations of motion of some Lagrangian from Λ_1^1: $a \in d_{_{E.L}}\Lambda_1^1(M)$ iff there exists Lagrangian L such that $a = d_{_{E.L}}L$.

From the 2-nd statement of Proposition 1 it follows that cohomology of this truncated complex is strictly related with de Rham cohomology:

$$H^0(\mathcal{C}^*, d_{_{E.L}}) = H^0(M)\,, H^1(\mathcal{C}^*, d_{_{E.L}}) = H^1(M) + \mathbf{R},\ H^2(\mathcal{C}^*, d_{_{E.L}}) = 0\,. \tag{2.11}$$

For our purposes it is useful to consider also the following modification of the complex (2.7). We consider the spaces $\{\overline{\Lambda^q}\}$ where $\overline{\Lambda^q} = \Lambda^q/\mathbf{R}$ if $q \geq 1$ and $\overline{\Lambda^0} = \Lambda^0 = \Omega^0(M)$. Elements of $\overline{\Lambda^q}$ $(q \geq 1)$ are q-Lagrangians which are defined up to constants. We denote by $\overline{L}$ the equivalence class of Lagrangian L in $\overline{\Lambda}$. One can consider instead the complex (2.7) the complex

$$\{\overline{\Lambda^q}(M), \overline{d}_{_{E.L}}\}: \quad \Lambda^0(M) \overset{\overline{d}_{_{E.L}}}{\longrightarrow} \overline{\Lambda^1}(M) \overset{\overline{d}_{_{E.L}}}{\longrightarrow} \overline{\Lambda^2}(M) \overset{\overline{d}_{_{E.L}}}{\longrightarrow} \dots\,. \tag{2.12}$$

and correspondingly the double complex $\{C^p(\mathcal{G}, \overline{\Lambda^q}), \overline{d}_{_{E.L}}, \overline{\delta}\}$ of p-cochains on $\mathcal{G}$ with values in $\overline{\Lambda^q}$. The differentials $\overline{d}_{_{E.L}}$ and $\overline{\delta}$ are correctly defined in a natural way: $\overline{d_{_{E.L}}\overline{\lambda}} \dot{=} \overline{d_{_{E.L}}\lambda}$ and $\overline{\delta}\overline{\lambda} = \overline{\delta\lambda}$ where $\overline{\lambda}$ is equivalence class in $C^*(\mathcal{G}, \overline{\Lambda^*})$ of the cochain λ in $C^*(\mathcal{G}, \Lambda^*)$. The differential $\overline{d}_{_{E.L}}$ does not differ essentially from $d_{_{E.L}}$: If λ is cochain with values in Lagrangians then it is easy to see that

$$\overline{d}_{_{E.L}}\overline{\lambda} = 0, \iff d_{_{E.L}}\lambda = 0 \tag{2.13}$$

To (2.10) corresponds the subcomplex

$$(\overline{\mathcal{C}^*}, d_{_{E.L}}): \qquad \Lambda^0(M) \overset{\overline{d}_{_{E.L}}}{\longrightarrow} \overline{\Lambda_1^1}(M) \overset{\overline{d}_{_{E.L}}}{\longrightarrow} (d_{_{E.L}}\overline{\Lambda_1^1}(M)) \longrightarrow 0 \tag{2.14}$$

of the complex (2.12). From (2.13) it follows that for the truncated complex $\overline{\mathcal{C}^*}$

$$H^0(\overline{\mathcal{C}^*}, d_{_{E.L}}) = H^0(M),\ H^1(\overline{\mathcal{C}^*}, d_{_{E.L}}) = H^1(M),\ H^2(\overline{\mathcal{C}^*}, d_{_{E.L}}) = 0\,. \tag{2.15}$$

We avoid here an appearance of non-pleasant constants like in (2.11). The difference between the complex $\{C^p(\mathcal{G}, \overline{\Lambda^q}), \overline{d}_{_{E.L}}, \overline{\delta}\}$ and the complex $\{C^p(\mathcal{G}, \Lambda^q), d_{_{E.L}}, \delta\}$ becomes non-trivial at least on the level of 1-cochains. It corresponds to the difference of time independent and time dependent Noether charges. (See for e.g. the Example 1 in the Section V.)

Finally we want to note that to every Lagrangian L on M corresponds the density A_L on the space $\hat{M} = M \times \{\text{space of parameters}\}$. (It is so called formalism where fields and space variables are on an equal footing [18]). To the functional (2.3) corresponds the integral of the density over the surface in $\hat{M}$ which is the graph of the map (2.2). For example to Lagrangian $L(q^\mu, \frac{dq^\mu}{dt})$ of classical mechanics one can correspond the density

$$A_L\left(q^\mu, \frac{dq^\mu}{d\tau}, \frac{dt}{d\tau}\right) = L\left(q^\mu, \frac{dq^\mu}{d\tau} / \frac{dt}{d\tau}\right) \cdot \frac{dt}{d\tau}; \quad \text{if } \tau \to \tau'(\tau), \text{ then } A_L \to \frac{d\tau}{d\tau'} A_L\,. \tag{2.16}$$

To a path $q^\mu(t)$ corresponds curve $(q^\mu(\tau), t(\tau))$ and $S_L([q(t)]) = S_{A_L}([q(\tau), t(\tau)])$ for any parametrization $q(\tau)$. It is easy to see that for densities A_L the difference between complexes (2.10) and (2.14) is removed. To redefinition of Lagrangian L on the constant c corresponds redefinition of A_L on the form cdt.)

III Cohomology of Lagrangians Double Complex and and its Spectral Sequences.

Now using the technique briefly described in the previous Section and in Appendix 2 we investigate systematically the problem which we considered in Introduction.

We study simultaneously two double complexes, the double complex $(E^{*.*}, d_{_{E.L}}, \delta)$ of cochains on $\mathcal{G}$ with values in the spaces of the complex $\mathcal{C}^*$ defined by (2.10), $\{E^{p.q}, d_{_{E.L}}, \delta\} = \{C^p(\mathcal{G}, \mathcal{C}^q), \delta, d_{_{E.L}})\}$ and the double complex $(\overline{E^{*.*}}, \overline{d}_{_{E.L}}, \overline{\delta})$ of cochains on $\mathcal{G}$ with values in the spaces of the complex $\overline{\mathcal{C}}^*$ defined by (2.14), $\{\overline{E^{p.q}}, \overline{d}_{_{E.L}}, \overline{\delta}\} = \{C^p(\mathcal{G}, \overline{\mathcal{C}^q}), \overline{d}_{_{E.L}}, \overline{\delta})\}$.

The complex $(E^{*.*}, d_{_{E.L}}, \delta)$ is represented by the following table

$$\begin{array}{ccccccc}
\Lambda^0(M) & \xrightarrow{d_{E.L}} & \Lambda^1_1(M) & \xrightarrow{d_{E.L}} & d_{_{E.L}}\Lambda^1_1(M) & \xrightarrow{d_{E.L}} & 0 \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
C^1(\mathcal{G}, \Lambda^0(M)) & \xrightarrow{d_{E.L}} & C^1(\mathcal{G}, \Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & C^1(\mathcal{G}, d_{_{E.L}}\Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & 0 \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
C^2(\mathcal{G}, \Lambda^0(M)) & \xrightarrow{d_{E.L}} & C^2(\mathcal{G}, \Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & C^2(\mathcal{G}, d_{_{E.L}}\Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & 0 \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\cdot & & \cdot & & \cdot & & \\
\cdot & & \cdot & & \cdot & & \\
\cdot & & \cdot & & \cdot & &
\end{array} \tag{3.1}$$

(The table represented the complex $(\overline{E^{*.*}}, d_{_{E.L}}, \delta)$ differs from (3.1) by putting the "bar"s in corresponding places.

The differential Q of the complex (3.1) is equal to

$$Q = (-1)^q \delta + d_{_{E.L}}, \text{ for the complex } E^{*.*} \tag{3.2}$$

and correspondingly $\overline{Q} = (-1)^q \overline{\delta} + \overline{d}_{_{E.L}}$ for the complex $\overline{E^{*.*}}$.

The problem of weakly invariant Lagrangians classification can be reformulated in terms of these double complexes.

For this purpose we consider their spectral sequences $\{E_r^{*.*}\}, \{\overline{E_r^{*.*}}\}$ and transposed spectral sequences $\{{}^tE_r^{*.*}\}, \{\overline{{}^tE_r^{*.*}}\}$.

The relations between $\{{}^tE_r^{*.*}\}$ and $\{E_r^{*.*}\}$ lead to the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians with time-independent Noether charges, the relations between $\{\overline{{}^tE_r^{*.*}}\}$ and $\{\overline{E_r^{*.*}}\}$ lead to the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians with time-dependent Noether charges and the relations between $\{E_r^{*.*}\}$ and $\{\overline{E_r^{*.*}}\}$ lead to the relations between these two hierarchies.

We denote by $\mathcal{V}_{0.0}$ (see Introduction) the subspace of weakly $\mathcal{G}$-invariant Lagrangians in the space $E^{0.1}$, i.e. Lagrangians whose motions equations l.h.s. are $\mathcal{G}$-invariant:

$$\mathcal{V}_{0.0} = \{L \colon L \in \Lambda^1_1 \text{ and } \delta d_{_{E.L}} L = 0\}\,. \tag{3.3}$$

One can see that the cochain $\mathbf{f} = (d_{_{E.L}} L, 0, 0)$ is the cocycle of differential Q iff $L \in \mathcal{V}_{0.0}$. The cohomology class $[(d_{_{E.L}} L, 0, 0)]$ of this cocycle belongs to $H^2(Q)$. If we express the cohomology of differential Q via the stable terms of transposed spectral sequence $\{ {}^t E_r^{*.*} \}$, i.e. calculating $H^*(Q)$ in perturbation theory, considering in (3.1) the differential δ as zeroth order approximation for the differential Q, we see that $[d_{_{E.L}} \mathcal{V}_{0.0}]_\infty = {}^t E_\infty^{0.2}$ is the subspace of $H^2(Q)$. On the other hand if we express the cohomology of differential Q via the stable terms of spectral sequence $\{E_r^{*.*}\}$, i.e. calculating $H^*(Q)$ in perturbation theory, considering in (3.1) the differential $d_{_{E.L}}$ as zeroth order approximation, we express $H^2(Q)$ in terms of $\{E_\infty^{p.2-p}\}$. The relations between the space ${}^t E_\infty^{0.2}$ and the spaces $\{E_\infty^{p.2-p}\}$ lead to the relations between the space of weakly $\mathcal{G}$-invariant Lagrangians and cohomologies groups of $\mathcal{G}$ and M.

The spaces $\{E_r^{p.q}\}$ and $\{\overline{E_r^{p.q}}\}$

We pay more attention on the calculations for the spaces $\{E_r^{*.*}\}$. The calculations for the spaces $\{\overline{E_r^{*.*}}\}$ can be performed analogously using (2.13).
The spaces $\{E_1^{p.q}\}$ are equal to the cohomologies of operator $d_{_{E.L}}$: $E_1^{p.q} = H(d_{_{E.L}}, E^{p.q})$. (See Appendix 2).

From (2.11) and (2.15) it immediately follows that

$$
\begin{array}{ccc}
 & E_1^{*.*} & \\
 & & \\
\mathbf{R} & H^1(M) \oplus \mathbf{R} & 0 \\
C^1(\mathcal{G}) & C^1(\mathcal{G}, H^1(M) \oplus \mathbf{R}) & 0 \\
C^2(\mathcal{G}) & \cdots & 0 \\
\cdots & \cdots & 0
\end{array}
\qquad
\begin{array}{ccc}
 & \overline{E_1^{*.*}} & \\
 & & \\
\mathbf{R} & H^1(M) & 0 \\
C^1(\mathcal{G}) & C^1(\mathcal{G}, H^1(M)) & 0 \\
C^2(\mathcal{G}) & \cdots & 0 \\
\cdots & \cdots & 0
\end{array}
\tag{3.4}
$$

Hereafter we identify the differential forms with Lagrangians corresponding to them by (2.4) and the differential $d_{_{E.L}}$ on these Lagrangians with differential d on forms.

In the columns of $E_1^{*.*}$ acts operator d_1 which is generated by δ and according to (A2.13) $E_2^{p.q} = H(E_1^{p.q}, d_1)$. It is easy to see that $E_2^{p.0} = H^p(\mathcal{G})$ is p-th cohomology group of the Lie algebra $\mathcal{G}$ with coefficients in $\mathbf{R}$.

Now we prove that $E_1^{0.1} = E_2^{0.1}$. Indeed if $c \in E_1^{0.1}$ is a constant ($c \in \mathbf{R}$) then $d_1 c$ is evidently equal to zero. To prove that $d_1 H^1(M) = 0$ we consider the following homomorphism π from the space of differential 1-forms into the space of 1-cochains on $\mathcal{G}$ with values in functions on M (the space $\Lambda^0(M)$):

$$(\pi w)(h) = w \rfloor \tilde{h}, \tag{3.5}$$

where $\tilde{h}$ is the fundamental vector field Φh corresponding to the element h of the Lie algebra $\mathcal{G}$ by (2.1).

From the standard formulae of differential geometry it follows that

$$\text{if } dw = 0 \text{ then } \delta \pi w = 0 \text{ and } \delta w = d \pi w. \tag{3.6}$$

Hence for the cohomology class $[w]$ in $H^1(M)$ $d_1[w] = [\delta w] = [d\pi w] = 0$ in $E_1^{1.1}$. Hence $Z_1^{0.1} = E_1^{0.1}$ and $E_1^{0.1} = E_2^{0.1}$ because $B_1^{0.1} = 0$.

Now we calculate $E_2^{1.1}$. If $[c]_1 \in E_1^{1.1}$ then

$$c = \sum_\lambda t^{(\lambda)} \otimes w^{(\lambda)} + t' + d\alpha \tag{3.7}$$

where t, t' belong to $C^1(\mathcal{G})$ (are constants), the set $\{w^{(\lambda)}\}$ of differential closed 1- forms constitutes a basis in the space $H^1(M)$ of 1-cohomology and α is some element from $E^{1.0}$. The straightforward calculations using (3.5, 3.6) give that

$$d_1[c]_1 = \sum_\lambda [\delta t^{(\lambda)} \otimes w^{(\lambda)} + \delta t' + d(\ldots)] = 0 \Rightarrow \delta t^{(\lambda)} = 0 \text{ and } \delta t' = 0\,. \tag{3.8}$$

On the other hand coboundaries in $E_1^{1.1}$ are equal to zero because $E_1^{0.1} = E_2^{0.1}$. Hence from eq. (3.7) it follows that $E_2^{1.1} = H^1(M) \otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G})$. (In the case of complex $\overline{E_1^{1.1}}$, t' in (3.7) is equal to zero and from (2.13) it follows that (3.8) holds also.)

We arrive at the following tables

$$\begin{array}{cccc} & E_2^{*.*} & & \\ & & & \\ \mathbf{R} & H^1(M)\oplus\mathbf{R} & 0 \\ H^1(\mathcal{G}) & H^1(\mathcal{G})\otimes H^1(M)\oplus\mathbf{R} & 0 \\ H^2(\mathcal{G}) & \cdots & 0 \\ H^3(\mathcal{G}) & \cdots & 0 \end{array} \qquad \begin{array}{ccc} & \overline{E_2^{*.*}} & \\ & & \\ \mathbf{R} & H^1(M) & 0 \\ H^1(\mathcal{G}) & H^1(\mathcal{G})\otimes H^1(M) & 0 \\ H^2(\mathcal{G}) & \cdots & 0 \\ H^3(\mathcal{G}) & \cdots & 0 \end{array} \tag{3.9}$$

One can show that the spaces $\{E_2^{p.q}\}$ in (3.9) which are of interest for us ($p+q \leq 2$) are stable: $E_2^{p.q} = E_3^{p.q} = \ldots = E_\infty^{p.q}$. (The same for $\{\overline{E_2^{p.q}}\}$.)

It is evident without any calculations for the spaces $E_2^{0.0}$, $E_2^{1.0}$ because differentials d_2 which acts on these spaces goes out of the table and the boundaries are zero by the same reasons. The spaces $E_2^{0.1}$ and $E_2^{2.0}$ are stable because the differential d_2 acting from the space $E_2^{0.1}$ into the space $E_2^{2.0}$ transforms it to zero. It follows from eq. (3.5): $d_2[w] = [Q(w, \pi w)] = [\delta \pi w] = 0$. The same arguments lead to the stability of the space $E_2^{1.1}$. One can perform the analogous considerations for the spaces $\{\overline{E_2^{p.q}}\}$.

Hence the tables (3.9) establish the relations between the spaces $H^m(Q)$, $H^m(\overline{Q})$ ($m = 0,1,2$) and the spaces $E_\infty^{p.q}$, $\overline{E_\infty^{p.m-p}}$ correspondingly, according to eq. (A2.11).

$H^0(Q) = H^0(\overline{Q}) = \mathbf{R}$. Considering the first "antidiagonal" $\{E_\infty^{0.1}, E_\infty^{1.0},\}$ in (3.9) we see from (A2.11) that

$$H^1(\mathcal{G}) \subseteq H^1(Q) \quad \text{and} \quad H^1(M) \oplus \mathbf{R} = H^1(Q)/H^1(\mathcal{G}). \tag{3.10}$$

These relations define canonical projection p_1 of $H^1(Q)$ on $H^1(M) \oplus \mathbf{R}$ and isomorphism ι_1 of $\mathbf{ker}p_1$ on $H^1(\mathcal{G})$: If $\mathbf{L} = (L, \alpha)$ is a cocycle of Q then $L = w + c$ where w is a closed form and c is a constant and $p_1([\mathbf{L}]) = [w] + c$. If $c = 0$ and $w = df$ then $\alpha - \delta f$ is 1-cocycle in constants which is equal to $\iota_1([\mathbf{L}])$.

Using the homomorphism (3.5) one can establish also the isomorphism

$$H^1(M) \oplus H^1(\mathcal{G}) \oplus \mathbf{R} \longrightarrow H^1(Q)\colon \quad [w] + t + c \longrightarrow [w + c, t + \pi w] \tag{3.11}$$

which corresponds to (3.10) and splits $H^1(Q)$ on components.

The analogous considerations for the table $\overline{E_2^{*.*}}$ lead to formulae analogous to (3.10, 3.11): $H^1(\mathcal{G}) \subseteq H^1(\overline{Q})$ and $H^1(M) = H^1(\overline{Q})/H^1(\mathcal{G})$; $H^1(M) \oplus H^1(\mathcal{G}) = H^1(\overline{Q})$.

Considering in the same way the second "antidiagonal" $\{E_\infty^{0.2}, E_\infty^{1.1}, E_\infty^{2.0}\}$ in (3.9) we see that

$$H^2(\mathcal{G}) \subseteq H^2(Q) \quad \text{and} \quad H^1(M) \otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G}) = H^2(Q)/H^2(\mathcal{G})\,. \tag{3.12}$$

These relations define canonical projection

$$p_2\colon \quad H^2(Q) \longrightarrow H^1(M)\otimes H^1(\mathcal{G}) + H^1(\mathcal{G}) \tag{3.13}$$

and on the kernel of p_2 the isomorphism

$$\iota_2\colon \quad \mathbf{ker} p_2 \longrightarrow H^2(\mathcal{G}). \tag{3.14}$$

We consider now (3.13) and (3.14) in components.

Let $\mathbf{f} = [\mathcal{F}, \lambda, f] \in H^2(Q)$ be a cohomology class of cocycle $(\mathcal{F}, \lambda, f)$: $Q(\mathcal{F}, \lambda, f) = 0$. $d_{E.L}\lambda = -\delta\mathcal{F}, \delta\lambda = df, \delta f = 0$. ($\mathcal{F} \in E^{0.2}, \lambda \in E^{1.1}, f \in E^{0.2}$). The space $E^{0.2}$ contains only coboundaries, so cocycle $(\mathcal{F}, \lambda, f)$ is cohomological to $(0, \lambda', f)$ where $\lambda' = \lambda - \delta L$ (L: $\mathcal{F} = d_{E.L}L$). $d_{E.L}\lambda' = 0$, so from Proposition 1 it follows that 1-cochain λ' takes values in closed differential 1-forms + constants:

$$\forall h \in \mathcal{G} \quad \lambda'(h) = w(h) + t(h). \tag{3.15}$$

Using (3.7,3.8) we see that to λ' corresponds element of $H^1(M)\otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G})$ which is nothing but $p_2(\mathbf{f})$.

In the case if $p_2(\mathbf{f}) = 0$ then it means that $\lambda' = d\alpha$ where $\alpha \in E^{1.0}$ and the cocycle $(0, \lambda', f)$ is cohomological to a cocycle $(0, 0, f - \delta\alpha)$. $d(f - \delta\alpha) = 0$ so $f - \delta\alpha$ is cocycle in $Z^2(\mathcal{G})$. The cohomology class of $f - \delta\alpha$ in $H^2(\mathcal{G})$ is nothing but $\iota_2(\mathbf{f})$.

The analogous considerations for the second "antidiagonal" in the table $\overline{E_2^{*.*}}$ lead to the analogous conclusions for $H^2(\overline{Q})$: $H^2(\mathcal{G}) \subseteq H^2(\overline{Q})$ and $H^1(M)\otimes H^1(\mathcal{G}) = H^2(\overline{Q})/H^2(\mathcal{G})$; $\overline{p_2}$: $H^2(\overline{Q}) \longrightarrow H^1(M)\otimes H^1(\mathcal{G})$. On the kernel of $\overline{p_2}$ is defined isomorphism $\overline{\iota_2}$: $\quad \mathbf{ker}\overline{p_2} \longrightarrow H^2(\mathcal{G})$.

From the considerations above we see that natural relations between complexes $(E^{*.*}, d_{E.L}, \delta), (\overline{E^{*.*}}, \overline{d}_{E.L}, \overline{\delta})$ lead to isomorphisms

$$H^1(Q) = H^1(\overline{Q}) \oplus \mathbf{R}, \quad H^2(Q) = H^2(\overline{Q}) \oplus H^1(\mathcal{G})\,. \tag{3.16}$$

The decomposition of $H^2(Q)$ defines the projection

$$\sigma\colon H^2(Q) \to H^1(\mathcal{G}). \tag{3.17}$$

$\sigma(\mathbf{f})$ is equal to the element of $H^1(\mathcal{G})$ in the r.h.s. of the eq. (3.15). This projection will be useful for extracting Lagrangians whose Noether charges are time independent in the space $\mathcal{V}_{0.0}$ of weakly invariant Lagrangians.

Now we return again to the complex (3.1) and express the cohomologies of $H(Q)$ and $H(\overline{Q})$ in terms of transposed spectral sequences $\{{}^tE_r^{*.*}\}$ and $\{\overline{{}^tE_r^{*.*}}\}$.

For constructing ${}^tE_1^{*.*}$ and $\{\overline{{}^tE_r^{*.*}}\}$ we have to consider as zeroth order approximation the cohomology of vertical differential δ: $\{{}^tE_1^{*.*}\} = H(E^{*.*}, \delta)$ and $\{\overline{{}^tE_1^{*.*}}\} = H(\overline{E^{*.*}}, \delta)$. We arrive at the tables

$$\begin{array}{ccc} & {}^tE_1^{*.*} & \\ & & \\ \Lambda^0_{inv} & \Lambda^1_{1inv} & d_{E.L}\mathcal{V}_{0.0} \\ H^1(\mathcal{G}, \Lambda^0(M)) & H^1(\mathcal{G}, \Lambda^1_1) & \cdots \\ H^2(\mathcal{G}, \Lambda^0(M)) & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array} \qquad \begin{array}{ccc} & \overline{{}^tE_1^{*.*}} & \\ & & \\ \Lambda^0_{inv} & \overline{\Lambda^1_{1inv}} & \overline{d}_{E.L}\overline{\mathcal{V}}_{0.0} \\ H^1(\mathcal{G}, \Lambda^0(M)) & H^1(\mathcal{G}, \overline{\Lambda^1_1}) & \cdots \\ H^2(\mathcal{G}, \Lambda^0(M)) & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array} \tag{3.18}$$

Here $\Lambda^0_{inv} = C^0(\mathcal{G}, \Lambda^0(M))$ is the space of the functions on M which are invariant under the action of the Lie algebra $\mathcal{G}$. The Λ^1_{1inv} is the space of $\mathcal{G}$-invariant Lagrangians from

Λ^1_1. The space $\overline{\Lambda^1_{1\,inv}}$ in the right table contains the classes (Lagrangians factorised by constants) whose variation under $\mathcal{G}$ symmetry transformations produces $\mathcal{G}$-cochain with values in constants: $\overline{\Lambda} \in \overline{\Lambda^1_{1\,inv}} \Leftrightarrow \overline{\delta\Lambda} = 0 \Leftrightarrow \delta_i\Lambda = t_i$. These Lagrangians have linear time dependent Noether charges (see (1.13)). The space $d_{_{E.L}}\mathcal{V}_{0.0}$ is the image under differential $d_{_{E.L}}$ of the subspace $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians (see (3.3)). From (2.13) it follows that $\overline{{}^tE^{0.2}} = \overline{d_{_{E.L}}\mathcal{V}_{0.0}}$ also.

The differential ${}^td_1$ which is generated by $d_{_{E.L}}$ acts in rows of the table ${}^tE_1^{*.*}$ (compare with the table (3.4)). For ${}^tE_2^{*.*} = H({}^tE_1^{*.*},\, {}^td_1)$ we obtain

$$
{}^tE_2^{*.*} = \begin{array}{ccc} \mathbf{R} & H^1_{inv}(M)\oplus\mathbf{R} & d_{_{E.L}}\mathcal{V}_{0.0}\big/(d_{_{E.L}}\Lambda^1_{1\,inv}) \\ {}^tE_2^{1.0} & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array} \tag{3.19}
$$

$H^1_{inv}(M)$ is the space of closed $\mathcal{G}$-invariant differential 1-forms factorised by the differentials of $\mathcal{G}$-invariant functions.

The analogous table one can consider for $\overline{{}^tE_2^{*.*}}$.

The space ${}^tE_2^{1.0}$ in (3.19) is the subspace of $H^1(\mathcal{G}, \Lambda^0(M))$. It contains the classes $[\alpha]$ from $H^1(\mathcal{G}, \Lambda^0(M))$ for whose the eq. $d\alpha = \delta L$ has the solution. (Compare with (1.2)). We see that the table (3.19) is not stable in the spaces which we are interesting in because the differential ${}^td_2$ acting from ${}^tE_2^{1.0}$ in ${}^tE_2^{0.2}$ is not trivial: ${}^td_2[\alpha] = {}^t[d_{_{E.L}}L]_2$. The next table ${}^tE_3^{*.*} = H({}^tE_2^{*.*},\, {}^td_2)$ is stable in the spaces which we are interesting in:

$$
{}^tE_3^{*.*} = \begin{array}{ccc} \mathbf{R} & H^1_{inv}(M)\oplus\mathbf{R} & \frac{d_{_{E.L}}\mathcal{V}_{0.0}\big/(d_{_{E.L}}\Lambda^1_{1\,inv})}{\mathbf{Im}({}^td_2\,{}^tE_2^{1.0})} \\ {}^tE_3^{0.1} & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array} \tag{3.20}
$$

From the general properties of spectral sequences it follows that in (3.20) ${}^tE_3^{0.2} = {}^tE_\infty^{0.2}$ *is the subspace in* $H^2(Q)$ and the space ${}^tE_3^{1.0} = {}^tE_\infty^{1.0}$ (which is the subspace of ${}^tE_2^{1.0}$) is the factorspace of $H^1(Q)$ by the space ${}^tE_3^{0.1} = H_{inv}(M)\oplus\mathbf{R}$ (compare with (3.10)). Hence from the decomposition (3.11) of $H^1(Q)$ it follows that

$$
{}^tE_3^{1.0} = (H^1(M)\oplus H^1(\mathcal{G}))\big/H^1_{inv}(M)\,. \tag{3.21}
$$

In (3.21) $H^1_{inv}(M)$ is considered as naturally embedded in $H^1(M)\oplus H^1(\mathcal{G})$. If $w \in H^1_{inv}(M)$ is trivial in $H^1(M)$ then $\delta f \in H^1(\mathcal{G})$ where $w = df$.

Performing the corresponding calculations for the table $\overline{{}^tE_3^{*.*}}$ one has to put the "bar"s in ${}^tE_3^{0.2}$, the space ${}^tE_3^{0.1}$ has to be changed on $\overline{{}^tE_3^{0.1}} = H^1(M)_{inv}$. The spaces ${}^tE_1^{1.0}$ and $\overline{{}^tE_1^{1.0}}$ as well as the spaces ${}^tE_3^{1.0}$ and $\overline{{}^tE_3^{1.0}}$ coincide but on the other hand ${}^tE_2^{1.0} \subseteq \overline{{}^tE_2^{1.0}}$.

In the tables (3.18–3.20) every space ${}^tE_r^{1.0}$ is the subspace of previous one and correspondingly every space ${}^tE_r^{0.2}$ is the factorspace of previous one. We denote by Π_r the homomorphism which put in correspondence to every weakly $\mathcal{G}$-invariant Lagrangian its equivalence class in the space ${}^tE_r^{0.2}$:

$$
\Pi_r\colon\quad \mathcal{V}_{0.0} \to {}^tE_r^{0.2}, \quad \Pi_r(L) = {}^t[d_{_{E.L}}L]_r. \quad \forall L \in \mathcal{V}_{0.0} \quad \mathbf{Im}\Pi_3 \subseteq H^2(Q). \tag{3.22}
$$

Analogously $\overline{\Pi}_r$: $\quad \overline{\mathcal{V}}_{0.0} \to \overline{{}^tE_r^{0.2}}$.

Comparing the content of the spaces $\{{}^tE_r^{1.0}\}$ and $\{{}^tE_r^{0.2}\}$ in the transposed spectral sequences (3.18–3.20) with the results above for spectral sequence $\{E_r^{*.*}\}$ we come to

Proposition 2

a) To weakly $\mathcal{G}$-invariant Lagrangians correspond elements in the space ${}^tE_3^{0.2}$, i.e. in $H^2(Q)$. Thus to these Lagrangians via homomorphisms p_2 and ι_2 (3.13,3.14) correspond elements in $E_2^{1.1}$ or in $E_2^{2.0}$.

b) To weakly $\mathcal{G}$-invariant Lagrangians whose image in the space ${}^tE_3^{0.2}$ is equal to zero: $\Pi_3(L) = 0$, correspond elements in ${}^tE_2^{0.2}$ which belong to the image of the differential ${}^td_2$. Thus to these Lagrangians correspond elements in ${}^tE_2^{1.0}$ which are defined up to the space ${}^tE_3^{1.0}$ defined by (3.21), which is the kernel of this differential.

c) The space ${}^tE_3^{1.0}$ is related with weakly $\mathcal{G}$-invariant Lagrangians whose image in the space ${}^tE_2^{0.2}$ is equal zero: $\Pi_2(L) = 0$.

The analogous statement is valid for the spaces $\{\overline{{}^tE_r^{*.*}}\}$.

In the next section using this Proposition we establish the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians.

IV The calculation of the hierarchy

Now using the calculations of the previous section for a given pair $[\mathcal{G}, M]$ we establish the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians.

Let $\mathcal{U}$ be an arbitrary subspace in the space $\Lambda_1^1(M)$ of the classical mechanics Lagrangians on M.

Let $\mathcal{U}_{0.0}$ be the subspace of weakly $\mathcal{G}$-invariant Lagrangians in $\mathcal{U}$: $\mathcal{U}_{0.0} = \mathcal{V}_{0.0} \cap \mathcal{U}$, where $\mathcal{V}_{0.0}$ is the subspace (3.3) of the all weakly $\mathcal{G}$-invariant Lagrangians in $\Lambda_1^1(M)$. From the Proposition 1 and (3.3) it follows that for an arbitrary L in $\mathcal{U}$ the condition that $L \in \mathcal{U}_{0.0}$ is equivalent to the condition that the cochain δL takes values in closed differential forms + constants:

$$\delta d_{_{E.L}} L = 0 \Leftrightarrow \delta_i L = w_{i\mu}\dot{q}^\mu + t_i \text{ and } dw_i = dt_i = 0\,. \qquad \text{(Compare with (1.11).)} \tag{4.1}$$

$\delta_i L$ is the value of the cochain δL on the basis vector e_i of the Lie algebra $\mathcal{G}$. (As always we identify differential forms with Lagrangians corresponding to them by (2.4))

Using the homomorphism Π_3 defined by (3.22) and the projection homomorphism (3.17) σ_2 of $H^2(Q)$ on $H^1(\mathcal{G})$ we consider the following composed homomorphism: $\Psi = \sigma \circ \Pi_3$: $\mathcal{U}_{0.0} \to H^2(Q) \to K_0 = H^1(\mathcal{G})$. In the components according to (3.15), $\Psi_i(L) = t_i$ where t_i is defined by (4.1). We denote by $\mathcal{U}_{0.1}$ the kernel of this homomorphism. In the case $\mathcal{U} = \Lambda_1^1(M)$ it is just the space $\mathcal{V}_{0.1}$ in (1.14) defined by the condition (1.12).

Now on the subspaces of $\mathcal{U}_{0.1}$ and on the subspaces of $\mathcal{U}_{0.0}$ using the Proposition 2 we define in the recurrent way the homomorphisms $\{\phi_s\}$ and correspondingly $\{\overline{\phi}_s\}$ such that every homomorphism is defined on the kernel of previous one. Moreover the definition spaces for these homomorphisms will be related via the homomorphism Ψ.

Using the statement a) of the Proposition 2 we consider the composed homomorphisms $\phi_1 = p_2 \circ \Pi_3$: $\mathcal{U}_{0.0} \to H^2(Q) \to H^1(M) \otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G})$ and $\overline{\phi}_1 = \overline{p}_2 \circ \overline{\Pi}_3$: $\mathcal{U}_{0.0} \to H^2(\overline{Q}) \to K_1 = H^1(M) \otimes H^1(\mathcal{G})$. From (3.15, 16) it follows that the restriction of $\overline{\phi}_1$ on

the subspace $\mathcal{U}_{0.1}$ coincides with ϕ_1. We denote by $\mathcal{U}_{1.0}$ the kernel of the homomorphism $\overline{\phi}_1$ and by $\mathcal{U}_{1.1}$ the kernel of the homomorphism ϕ_1. $\mathcal{U}_{1.1}$ is also the kernel of homomorphism Ψ restricted on $\mathcal{U}_{1.0}$. On the spaces $\mathcal{U}_{1.1}$ and $\mathcal{U}_{1.0}$ one can consider composed homomorphisms $\phi_2 = \iota_2 \circ \Pi_3 : \mathcal{U}_{1.1} \to H^2(Q) \to K_2 = H^2(\mathcal{G})$ and $\overline{\phi}_2 = \overline{\iota}_2 \circ \overline{\Pi}_3 : \mathcal{U}_{1.0} \to H^2(\overline{Q}) \to K_2$ correspondingly. $\overline{\phi}_2$ evidently coincides with ϕ_2 on $\mathcal{U}_{1.1}$.

For example if for Lagrangian L in $\mathcal{U}$ the condition (4.1) is satisfied, i.e. $L \in \mathcal{U}_{0.0}$, then $\overline{\phi}_1(L)$ is equal to the cohomology class of $w_{i\mu}dq^\mu$ in $H^1(\mathcal{G}) \otimes H^1(M)$ defined by (3.7); $L \in \mathcal{U}_{1.0}$ iff $\{w_{i\mu}dq^\mu\}$ are exact forms. In this case $\overline{\phi}_2(L)$ is equal to the cohomology class in $H^2(\mathcal{G})$ of the cocycle $f_{ij} = (\delta\alpha)_{ij}$ where $d\alpha_i = w_i$. If $t_i = 0$ also then $L \in \mathcal{U}_{1.1}$.

We denote by $\mathcal{U}_{2.0}$ the kernel of homomorphism $\overline{\phi}_2$ and by $\mathcal{U}_{2.1}$ the kernel of homomorphism ϕ_2. It is easy to see that $\mathcal{U}_{2.1} = \mathbf{ker}\Psi|_{\mathcal{U}_{2.0}}$.

For every Lagrangian $L \in \mathcal{U}_{2.1}$, $\Pi_3(L) = 0$. From the statement b) of the Proposition 2 it follows that one can consider the composed homomorphism

$$\phi_3 = ({}^t d_2)^{-1} \circ \Pi_2 : \mathcal{U}_{2.1} \to {}^t E_2^{0.2} \to K_3 = \frac{H^1(\mathcal{G}, \Lambda^0(M))}{(H^1(M) \oplus H^1(\mathcal{G}))/H^1_{inv}(M)}$$

Performing the analogous considerations for the space $\mathcal{U}_{2.0}$ we can consider the composed homomorphism $\overline{\phi}_3 = (\overline{{}^t d_2})^{-1} \circ \overline{\Pi}_2 : \mathcal{U}_{2.0} \to {}^t E_2^{0.2} \to K_3$.

One can see that in this case as in the previous ones, $\overline{\phi}_3|_{\mathcal{U}_{2.1}} = \phi_3$ and $\mathcal{U}_{3.1} = \mathbf{ker}\Psi|_{\mathcal{U}_{3.0}}$ also, where we denote by $\mathcal{U}_{3.1}$, $\mathcal{U}_{3.0}$ the kernels of ϕ_3 and $\overline{\phi}_3$ correspondingly.

For example in the case if L in (4.1) belongs to $\mathcal{U}_{2.1}$ then one can choose α_i such that $da_i = w_{i\mu}dx^\mu$ and $(\delta\alpha)_{ij} = 0$ because $\overline{\phi}_2(L) = 0$. The equivalence class of α_i in K_3 is $\overline{\phi}_3(L)$.

In the case if $L \in \mathcal{U}_{3.1}$ then $\Pi_2(L) = 0$. It means that the value of the homomorphism Π_1 (see 3.22) on this Lagrangian is equal to the value of this homomorphism on some $\mathcal{G}$-invariant Lagrangian: $\Pi_1(L) = d_{_{E.L}}L = d_{_{E.L}}L_{inv}$. From Proposition 1 it follows that $L = L_{inv} + w$ where closed differential 1-form w is defined uniquely up to closed $\mathcal{G}$-invariant form and exact form. This defines the homomorphism $\phi_4(L)$: $\mathcal{U}_{3.1} \to K_4 = H^1(M)/(H^1_{inv(M)})_*$ where $(H^1_{inv(M)})_*$ is the image of $H^1_{inv(M)}$ in $H^1(M)$ under the canonical homomorphism. More formally ϕ_4 can be defined as composed homomorphism with values in the kernel ${}^t E_3^{0.1}$ (3.21) of the differential ${}^t d_2$ (see the statement c) of the Proposition 2): On the space $\mathcal{U}_{3.1}$ the image of Π_1 belongs to the image of differential ${}^t d_1$ acting on the space ${}^t E_1^{0.1}$ in the table (3.18), hence $\phi_4 = \pi \circ (\mathbf{id} - ({}^t d_1)^{-1} d_{_{E.L}})$: $\mathcal{U}_{3.1} \to E^{0.1} \to K_4 \subseteq {}^t E_3^{1.0}$ where π is defined by (3.5).

Analogously one can define the homomorphism $\overline{\phi}_4(L)$: $\mathcal{U}_{3.0} \to K_4$.

Similarly to previous cases $\overline{\phi}_4|_{\mathcal{U}_{3.1}} = \phi_4$ and $\mathcal{U}_{4.1} = \mathbf{ker}\Psi|_{\mathcal{U}_{4.0}}$ also, where we denote by $\mathcal{U}_{4.1}$, $\mathcal{U}_{4.0}$ the kernels of ϕ_3 and $\overline{\phi}_3$ correspondingly.

From the definitions of ϕ_4 and $\overline{\phi}_4$ it is evident that Lagrangians belonging to $\mathcal{U}_{4.1}$ can be reduced to $\mathcal{G}$-invariant by the redefinition on exact form (full derivative.)

The spaces $\{\mathcal{U}_{s.1}, \mathcal{U}_{s.0}\}$ constructed here coincide with the spaces $\{\mathcal{V}_{s.1}, \mathcal{V}_{s.0}\}$ considered in the Introduction. (see (1.14, 1.15)) in the case if $\mathcal{U} = \Lambda^1_1(M)$.

These considerations can be summarized in the

Theorem. Let $\mathcal{U}$ be an arbitrary subspace in the space of classical mechanics Lagrangians for a given $[\mathcal{G}, M]$ pair. Let $\mathcal{U}_{0.0}$ be the subspace of $\mathcal{U}$ defined by (4.1) which contains the weakly $\mathcal{G}$-invariant Lagrangians in $\mathcal{U}$. Then the following relations which

establish the classification (hierarchy) in the space $\mathcal{U}_{0.0}$ are satisfied

$$\begin{array}{ccccccccc}
 & & \mathcal{U}_{4.1} & \subseteq & \mathcal{U}_{4.0} & & & & \\
 & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \\
K_4 & \xleftarrow{\phi_4} & \mathcal{U}_{3.1} & \subseteq & \mathcal{U}_{3.0} & \xrightarrow{\overline{\phi}_4} & K_4 & = & H^1(M)\big/(H^1_{inv}(M))_* \\
 & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \\
K_3 & \xleftarrow{\phi_3} & \mathcal{U}_{2.1} & \subseteq & \mathcal{U}_{2.0} & \xrightarrow{\overline{\phi}_3} & K_3 & = & \frac{H^1(\mathcal{G},\Lambda^0(M))}{(H^1(M\oplus H^1(\mathcal{G}))\big/H_{inv}(M)} \\
 & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \\
K_2 & \xleftarrow{\phi_2} & \mathcal{U}_{1.1} & \subseteq & \mathcal{U}_{1.0} & \xrightarrow{\overline{\phi}_2} & K_2 & = & H^2(\mathcal{G}) \\
 & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \\
K_1 & \xleftarrow{\phi_1} & \mathcal{U}_{0.1} & \subseteq & \mathcal{U}_{0.0} & \xrightarrow{\overline{\phi}_1} & K_1 & = & H^1(M)\otimes H^1(\mathcal{G}) \\
 & & & & \Psi\downarrow & & & & \\
 & & & & K_0 = H^1(\mathcal{G}) & & & &
\end{array} \tag{4.2}$$

The spaces $\mathcal{U}_{s.\sigma}$ are intersections of the space $\mathcal{U}$ with the spaces $\mathcal{V}_{s.\sigma}$ defined in Introduction (see 1.6–1.15); the double filtration $\{\mathcal{U}_{s.\sigma}\}$ is subordinated to the homomorphisms $\{\overline{\phi}_s, \phi_s, \Psi\}$ constructed above:

$$\mathcal{U}_{s.0} = \mathbf{ker}(\overline{\phi}_s\colon \mathcal{U}_{s-1.0} \to K_s), \quad \mathcal{U}_{s.1} = \mathbf{ker}(\phi_s\colon \mathcal{U}_{s-1.1} \to K_s),$$
$$\mathcal{U}_{s.1} = \mathbf{ker}(\Psi\colon \mathcal{U}_{s.0} \to K_0), \quad \overline{\phi}_s\big|_{\mathcal{U}_{s-1.1}} = \phi_s.$$

We denote the diagram (4.2) by $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ and call it the hierarchy diagram for the subspace $\mathcal{U}$. In the case if $\mathcal{U} = \Lambda^1_1(M)$ is the space of all Lagrangians of classical mechanics on M we denote the diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ shortly by $\mathcal{D}([\mathcal{G}, M])$.

The diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ measures the differences in the spaces $\{\mathcal{U}_{s.\sigma}\}$ for an arbitrary subspace $\mathcal{U}$.

We say that weakly $\mathcal{G}$-invariant Lagrangian $L \in \mathcal{U}$ is on the floor "s" if $L \in \mathcal{U}_{s.0}$ and $L \notin \mathcal{U}_{s+1.0}$. (All Lagrangians from $\mathcal{U}_{4.0}$ are on the 4-th floor.)

We say that weakly $\mathcal{G}$-invariant Lagrangian L is on the floor "s_+" if this Lagrangian is on the floor "s" and it belongs to $\mathcal{U}_{s.1}$. All other Lagrangians from the floor "s" are on the floor "s_-".

All Lagrangians which are on the " $+$ "-th floors have time independent Noether charges, except Lagrangians in zeroth floor.

The Lagrangians which are on the floor "s" have non-trivial image in the space K_{s+1} in (4.2). The Lagrangian on floor "s_-" have also non-trivial image in K_0 under homomorphism Ψ.

Returning to the table (1.6) in Introduction we can conclude that a Lagrangian which possesses the property "s" in (1.6) and which does not possesses the property "$s+1$" in (1.6) does have non-trivial image in the space K_{s+1}.

The evident but important corollary of the hierarchy diagram is that the floor is empty if the corresponding space K_s is trivial. For example in the case if the first de Rham cohomology of configuration space are trivial then $K_1 = K_4 = 0$ and the zeroth floor and the third floors are empty. In the case if the algebra $\mathcal{G}$ is semisimple only the floors $2_+, 3_+, 4_+$ can be non empty, because in this case $H^1(\mathcal{G}) = H^2(\mathcal{G}) = 0$, hence $K_0 = K_1 = K_2 = 0$.

The hierarchy diagram will be called trivial if all the spaces K_s are equal to zero.

In general the inverse statement is not valid. From the fact that the space K_s is not trivial does not follow that the floor "$s-1$" is not empty, because the homomorphisms in (4.2) are not in general surjective. For example homomorphism ϕ_2 in general is not surjective because the map ${}^t d_2$ which induces this homomorphism is defined on the subspace ${}^t E_2^{0.1}$ of the space $H^1(\mathcal{G}, \Lambda^0(M))$.

We say that the diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ is *full* on the floor "s_+" ($s < 4$) if ϕ_{s+1} is homomorphism onto the space K_s (surjective homomorphism), we say that this diagram is full on the floor "s_-"- if the restriction of Ψ on $\mathcal{U}_{s.1}$ is the surjective homomorphism. In the case if the diagram is full on the floors "s_+" and "s_-" we say that it is full on the floor "s".

For a given pair $[\mathcal{G}, M]$ two subspaces $\mathcal{U}$ and $\mathcal{U}'$ in the space $\Lambda^1_1(M)$ of classical mechanics Lagrangians on M will be called equivalent with respect to the hierarchy if the images of all the homomorphisms $\{\phi_s, \overline{\phi}_s, \Psi|_{\mathcal{U}_{s.1}}\}$ for the diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U}')$ coincide with the images of corresponding homomorphisms for the diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$. It is evident that in this case for arbitrary $L \in \mathcal{U}$ there exists $L' \in \mathcal{U}'$ such that $L' - L$ belongs to the space $\mathcal{U}_{4.1}$, i.e.

$$L' = L + L_{inv} + \text{full derivative}. \tag{4.3}$$

This construction can be used for defining in the space $\mathcal{U}_{0.0}$ a gradation corresponding to the filtration (4.2) (See the examples in the next Section.)

Now we use it for simplifying the diagram (4.2) for physically important subspace $\mathcal{U}^{pol}$ of Lagrangians which are polynomial in velocities. Let $\mathcal{U}^f = \Omega^1(M)$ be a subspace of formal Lagrangians in $\mathcal{U}^{pol}$ which correspond to differential forms by (2.4), and $\mathcal{U}^{sc} = \Lambda^0(M)$ be a subspace of formal Lagrangians in $\mathcal{U}^{pol}$ which are functions on M.

One can see that the space $\mathcal{U}^{pol}$ is equivalent to the space $\mathcal{U}^f \oplus \mathcal{U}^{sc}$ with respect to the hierarchy.

To prove it we note that every L in $\mathcal{U}^{pol}$ can be represented as

$$L(q, \dot q) = \sum_{n \geq 0} L_n(q, \dot q) = \sum_{n \geq 2} L_n(q, \dot q) + A_\mu(q)\dot q^\mu + \varphi(q)\,. \tag{4.4}$$

where $L_n(q, \dot q)$ is the polynom of $\dot q$ of the order n. Using the fact that the Lie derivative does not change the order of polynom ($(\delta L)_n = \delta(L_n)$) one can see that for homomorphism Ψ are responsible the functions on M and for homomorphisms $\phi_s, \overline{\phi}_s$ are responsible polynoms which are linear by velocities, i.e. differential 1-forms: $\Psi(L) = \Psi(\varphi), \phi_s(L) = \overline{\phi}_s(L) = \phi_s(A_\mu \dot q^\mu)$. This proves the equivalence.

The homomorphism Ψ takes values in the subspace of $H^1(\mathcal{G})$ which is isomorphic to the cohomologies of $H^1_{inv(M)}$ which are trivial in $H^1(M)$: If $\delta\varphi \in H^1(\mathcal{G})$ then $d\varphi \in H^1_{inv}(M)$, if $w \in H^1_{inv}(M)$ and $w = d\varphi$ then $\delta\varphi \in H^1(\mathcal{G})$.

From these facts it follows that for the diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U}^{pol})$ the following additional relations are satisfied:

$$\mathcal{U}^{pol}_{s.0} = \mathcal{U}^{pol}_{s.1} \oplus B, \quad \mathcal{U}^{pol}_{s.1} = \mathcal{U}_{4.1} \oplus A_s. \tag{4.5}$$

Here $B = \mathcal{U}^{sc}_{0.0}/\mathcal{U}^{sc}_{0.1}$ is the factorspace of functions in $\Lambda^0(M)$ whose $\mathcal{G}$-symmetry variation is constant by the space $\Lambda^0_{inv}(M)$ of $\mathcal{G}$-invariant functions. Correspondingly $A_s = \mathcal{U}^f_{s.1}$ are the subspaces of the space $\Omega^1(M)$ of 1-differential forms.

Weakly $\mathcal{G}$-invariant Lagrangians which belong to the space $\mathcal{U}^{pol}$ differ from the Lagrangians in $\mathcal{U}^{pol}_{4.1}$ ($\mathcal{G}$-invariant Lagrangians up to a full derivative) on the interaction with "electromagnetic field" whose field strength is $\mathcal{G}$-invariant. In particular a Lagrangian which is on the floor "s_-" differs from Lagrangian which is on the floor "s_+" on the interaction with "electric" field"–1-form $E_\mu = \partial\varphi/\partial q^\mu$. The value of this 1-form on every symmetries vector field is constant: $E_\mu(q)e^\mu_i(q) = t_i$ where $\{e^\mu_i(q)\}$ are fundamental vector fields corresponding to the basis $\{e_i\}$ in Lie algebra $\mathcal{G}$ via the map (2.1). The time dependence of corresponding Noether charge is proportional to t_i.

In general for an arbitrary Lagrangian in $\mathcal{V}$ these properties are not satisfied. (See e.g. Example 1 in the Section 5.)

The second physically important example of the subspace is the subspace $\mathcal{U}^{dens}$ of Lagrangians on M which are *densities*. (See the Remark in the 2-nd Section).

It is easy to see that in this case $\mathcal{U}_{s.1} = \mathcal{U}_{s.0}$, i.e. all the floors "$s_-$" are empty, because the homomorphism Ψ is trivial. (See the end of the 2-nd Section).

We do not consider here systematically the general methods to handle with calculations of the spaces K_s and corresponding homomorphisms for an arbitrary pair $[\mathcal{G}, M]$, but we note only some points which can be useful for analyzing the content of the space K_3 in the hierarchy diagram and the groups $H^1(\mathcal{G}, \Lambda^0(M))$ which generate these spaces.

First we note that the basic example of the $[\mathcal{G}, M]$ pair is provided with the following construction. Let $M \subseteq N$ be the subspace of a space N and the action of a Lie group G is defined on N. The action of G on N defines the pair $[\mathcal{G}, N]$ as well as the pair $[\mathcal{G}, M]$ where $\mathcal{G} = \mathcal{G}(G)$ is the Lie algebra of the group G. This pair in general cannot be generated by the action of a group on M.

We say that the pair $[\mathcal{G}, M]$ is transitive if fundamental vector fields span the tangent bundle TM: $\forall q \in M \quad \mathbf{Im}\Phi\big|_q = T_qM$. ($\Phi$ defines the action of $\mathcal{G}$ on M by (2.1).)

For example it is the case if the action of Lie algebra $\mathcal{G}$ on M is generated by transitive action of Lie group.

For a given $[\mathcal{G}, M]$ we can consider the stability subalgebra $\mathcal{G}_{st}(q)$ for every point $q \in M$: $\mathcal{G}_{st}(q) = \{\mathcal{G} \ni x \colon \Phi(x)|_q = 0\}$. In the case if the pair $[\mathcal{G}, M]$ is generated by the action of a group G, $\mathcal{G}_{st}(q)$ is isomorphic to the Lie algebra of stability subgroup of any point q_0.

Let $[\mathcal{G}, M]$ be a transitive pair. (The constructions below can be generalized on non-transitive case also).

If α is cocycle representing the cohomology class in $H^1(\mathcal{G}, \Lambda^0(M))$ then at arbitrary point q_0 it vanishes on the vectors in commutant $[\mathcal{G}_{st}(q_0), \mathcal{G}_{st}(q_0)]$. If this cocycle is generated by one-form w via homomorphism π, defined by (3.5) ($\alpha = \pi w$) then it vanishes at arbitrary point q_0 on all the vectors in $\mathcal{G}_{st}(q_0)$. Moreover πw is a coboundary iff w is coboundary. Thus for any point $q \in M$ one can consider homomorphisms

$$H^1(M) \overset{[\pi]}{\hookrightarrow} H^1(\mathcal{G}, \Lambda^0(M)) \xrightarrow{\rho_q} H^1(\mathcal{G}_{st}(q_0)),\ [\pi] \text{ is the injection and } \rho_q \circ [\pi] = 0,\ . \qquad (4.6)$$

In the case if the pair $[\mathcal{G}, M]$ is generated by transitive action of Lie group G (on N: $M \subseteq N$), then the image of the injection $[\pi]$ coincides with the kernel of ρ_q for an arbitrary point q, because the homomorphisms ρ_q for different points q are related by the adjoint action of the group transformation:

$$\forall(q,q_0),\ \forall\xi\in\mathcal{G}_{st}(q_0)\ \alpha(q,\mathbf{Ad}_g\xi)=\alpha(q_0,\xi)\text{ if }q=g\circ q_0. \tag{4.7}$$

Hence in this case K_3 can be injected in the factorspace of $H^1\mathcal{G}_{st}(q)$ for any q:

$$K_3\subseteq\mathbf{Im}\rho_q/\mathbf{Im}\rho_q\big|_{H^1(\mathcal{G})} \tag{4.8}$$

It gives the upper estimation for the dimension of the space K_3. (To calculate (4.8) it is useful to note that from definition of the space K_3, (3.21) and (3.5) it follows that the elements of K_3 are cocycles in $Z^1(\mathcal{G},\Lambda^0(M))$ factorized by cocycles which can be representing in the form: $\alpha=\pi w+t$, where w is closed 1-form and $t\in H^1(\mathcal{G})$.)

One can say more in the case if the pair $[\mathcal{G},M]$ is generated by the transitive action of the compact connected Lie group on the same space M. In this case taking the average of the group action on cocycle one comes to the injective homomorphism of $H^1(\mathcal{G},\Lambda^0(M))$ in $H^1(\mathcal{G})$:

$$\delta\alpha=0\Rightarrow\frac{1}{Vol(G)}\int\alpha^g d\mu_G=\bar{\alpha}:\quad H^1(\mathcal{G},\Lambda^0(M))\hookrightarrow H^1(\mathcal{G})\,. \tag{4.9}$$

($d\mu_G$ is invariant measure on G.)

For example if the pair $[\mathcal{G},M]$ is transitive and it is generated by the action of semisimple compact connected Lie group on the space M then using Whitehead lemmas ($H^1(\mathcal{G})=H^2(\mathcal{G})=0$), (4.8, 4.9) we see that all K_s are equal to zero and the hierarchy diagram is trivial.

The analogous conclusions can be made too in the case if the action is not transitive.

V Examples

In this Section using the hierarchy diagram (4.2) and considerations below we consider some examples of weakly $\mathcal{G}$-invariant Lagrangians classification.

Example 1

This example is the model example. But here we describe in details how to use the construction (4.3) for establishing gradation corresponding to the hierarchy filtration (4.2).

We consider the following pair $[\mathcal{G},M]$. Let $\mathcal{G}$ be Lie algebra ℓ_3 with generators e_1,e_2,e_3 such that $[e_1,e_2]=e_3,[e_2,e_3]=[e_3,e_1]=0$. Let a configuration space M be cylinder: $M=\mathbf{R}\times S^1$ with coordinates (z,φ). The homomorphism Φ (see 2.1) is defined by the relations

$$\Phi e_1=\tilde{e}_1=\frac{\partial}{\partial z},\quad \Phi e_2=\tilde{e}_2=z\frac{\partial}{\partial\varphi},\quad \Phi e_3=\tilde{e}_3=\frac{\partial}{\partial\varphi}\,. \tag{5.1}$$

This defines the pair $[\ell_3,S^1\times\mathbf{R}]$. For this pair first we calculate the hierarchy diagram $\mathcal{D}([\ell_3,S^1\times\mathbf{R}])$. We consider as $\mathcal{U}$ the whole space $\Lambda^1_1(M)$. From (5.1) it follows that every ℓ_3-invariant Lagrangian has the form $F(\dot{z})$ where F is an arbitrary function.

Now we calculate the spaces $\{K_s\}$. $K_0=H^1(\mathcal{G})=\mathbf{R}^2$ is generated by the cochains e^1 and e^2 ($\{e^i\}$ are dual to $\{e_j\}$: $e^i(e_j)=\delta^i_j$). The elements from $H^1(\mathcal{G})$ in components are $t_i=(a,b,0)$. $H^1(M)=\mathbf{R}$ is generated by 1-form $d\varphi$. Hence $K_1=\mathbf{R^2}$ is generated by cochains $(d\varphi,0,0)$ and $(0,d\varphi,0)$. $K_2=H^2(\mathcal{G})=\mathbf{R}^2$: the cocycles f_{ij} such that $f_{12}=0$

represent its cohomology class. It is easy to see that $H^1_{inv}(M) = \mathbf{R}$ is generated by 1-form dz. The stability subalgebra in every point (z,φ) is generated by the vector $e_2 - ze_3$, hence from (4.7) and the result for $H^1(\mathcal{G})$ it follows that $K_3 = 0$. (The explicit calculations without (4.7) give that $H^1(\mathcal{G}, \Lambda^0(M)) = \mathbf{R}^2$ is generated by the cocycles $\alpha_i = (0, az+b, a)$; $d(0, az+b, a) = (0, adz, 0) = \delta_i ad\varphi$, hence ${}^tE_3^{1.0} = {}^tE_2^{1.0} = H^1(\mathcal{G}, \Lambda^0(M))$ and $K_3 = 0$.)

The space $K_4 = \mathbf{R}$ is generated by the form $d\varphi$. We come to the result

$$K_0 = K_1 = K_2 = \mathbf{R}^2, \quad K_3 = 0, \quad K_4 = \mathbf{R}. \tag{5.2}$$

We already see that second floor of $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$ is empty.

The special analyze of the homomorphism ϕ_2 leads to the fact that 1-st floor is empty too: the image of ϕ_2 in K_2 is trivial because in this special case the subspaces ${}^tE_\infty^{0.2}$ and $E_\infty^{2.0}$ of $H^2(Q)$ have zero intersection.

Now we show that the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$ is full on the all floors except the second one and study the content of the spaces $\{\mathcal{V}_{s.1}, \mathcal{V}_{s.0}\}$.

For this purpose we consider the following 5-dimensional subspace of formal Lagrangians on $S^1 \times \mathbf{R}$:

$$U = \{L\colon \quad L = a\dot\varphi + bz\dot\varphi + cz + d\frac{\dot\varphi}{\dot z} + \frac{q}{2}\frac{\dot\varphi^2}{\dot z}\}, \tag{5.3}$$

where (a,b,c,d,q) are constants.

We show that the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], U)$ is full on all the floors except the first one. From this fact and from the emptiness of first floor for the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$ it follows that the whole space $\mathcal{V}$ of classical mechanics Lagrangians on M is equivalent to its subspace U with respect to the hierarchy. (See (4.3).)

The straightforward calculations give that for arbitrary Lagrangian from U

$$\delta_1 L = \mathcal{L}_{\frac{\partial}{\partial z}} L = bd\varphi + c,\ \delta_2 L = \mathcal{L}_{z\frac{\partial}{\partial\varphi}} L = adz + bzdz + d + qd\varphi,\ \delta_3 L = \mathcal{L}_{\frac{\partial}{\partial\varphi}} L = 0\,. \tag{5.4}$$

Comparing (5,4) with (4.1) we see that $U = U_{0.0}$.

Calculate the homomorphisms $\{\Psi, \phi_s, \overline{\phi}_s\}$ for the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], U)$ using (5.2–5.4). $\phi_2 = \overline{\phi}_2 = \phi_3 = \overline{\phi}_3 = 0$. $\forall L \in U, \Psi(L) = (c, d, 0) \in K_0$. If $c = d = 0$ then $L \in U_{0.1}$. $\phi_1(L) = \overline{\phi}_1(L) = (bd\phi, qd\varphi, 0) \in K_1$. If $b = q = 0$ then $L \in U_{1.0}$ and if $c = d = b = 0$ then $L \in U_{1.1}$. Hence $U_{3.0} = U_{2.0} = U_{1.0}$ and correspondingly $U_{3.1} = U_{2.1} = U_{1.0}$. $\phi_4(L) = \overline{\phi}_4(L) = ad\varphi \in K_4$. If $a = b = q = 0$ then we come to $U_{4.0}$. If also $c = d = 0$ then we come to $U_{4.1} = 0$.

All these homomorphisms except $\phi_2, \overline{\phi}_2$ are surjective. Hence the space $\Lambda^1_1(M)$ is reduced to its subspace U with respect to the hierarchy. Moreover these homomorphisms are injective on corresponding factor spaces. ($\mathbf{Im}\Psi = U_{0.0}/U_{1.0}$, $\mathbf{Im}\phi_s = U_{s-1.1}/U_{s.1}$ and $\mathbf{Im}\overline{\phi}_s = U_{s-1.0}/U_{s.0}$ if $s \neq 2$.)

From these considerations and (4.3) it follows that for every weakly ℓ_3-invariant Lagrangian there exists unique Lagrangian in U such that their difference belongs to $\mathcal{V}_{4.1}$:

$$\forall L \in \mathcal{V}_{0.0}\ \exists!(a,b,c,d,q)\colon L = F(\dot z) + \text{full derivative} + a\dot\varphi + bz\dot\varphi + cz + d\frac{\dot\varphi}{\dot z} + \frac{q}{2}\frac{\dot\varphi^2}{\dot z}\,. \tag{5.5}$$

Finally we come to the following gradation in the space $\mathcal{V}_{0.0}$ of weakly ℓ_3-invariant Lagrangians on $S^1 \times \mathbf{R}$:

$$\begin{gathered}\mathcal{V}_{3.1} = \mathcal{V}_{2.1} = \mathcal{V}_{1.1} = \mathcal{V}_{4.1} \oplus K_4 = \mathcal{V}_{4.1} \oplus \mathbf{R}, \\ \mathcal{V}_{0.1} = \mathcal{V}_{1.1} \oplus K_1 = \mathcal{V}_{1.1} \oplus \mathbf{R}^2,\ \mathcal{V}_{s.0} = \mathcal{V}_{s.1} \oplus K_0 = \mathcal{V}_{s.1} \oplus \mathbf{R}^2\,. \end{gathered} \tag{5.6}$$

We consider also briefly the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], \mathcal{U}^{pol})$ where $\mathcal{U}^{pol}$ is the subspace of Lagrangians which are polynomial in velocities. (See the end of 4-th Section.) It is easy to see that $\mathcal{U}^{pol}$ is reduced to the three-dimensional space U^{pol} which is the subspace of U defined by the additional conditions $d = q = 0$ in (5.3). The diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], \mathcal{U}^{pol})$ is not full on all the floors $\{s_-\}$ and on the floors 0_+ and 1_+. (One can show that in this case $\mathbf{Im}\Psi = \mathbf{R} \neq K_0, \mathbf{Im}\phi_1 = \mathbf{R} \neq K_1$). The space $\mathcal{U}^{pol}_{0.0}$ is parametrized by three-dimensional space U^{pol} (up to $\mathcal{U}^{pol}_{4.1}$) analogously to (5.5, 5.6) with conditions $d = q = 0$.

We note that in (5.5) the term $d(\dot\varphi/\dot z)$ which is responsible for time dependent Noether charges cannot be considered as interaction with "electric field" as in the case of Lagrangians in $\mathcal{U}^{pol}$.

We want to note also that all the considerations which lead to the formula (5.6) (except the property of homomorphism ϕ_2) where based on general relations which are established by the diagram (4.2).

Example 2

Let $M = \mathbf{R}^n$ be an n-dimensional linear space which acts on itself by translations. It defines the pair $[\mathbf{R}^n, \mathbf{R}^n]$. (We identify the affine space with corresponding linear space and with abelian algebra of translations.) It is easy to see that $K_0 = \mathbf{R}^n, K_2 = \mathbf{R}^n \wedge \mathbf{R}^n, K_1 = K_3 = K_4 = 0$. The space of Lagrangians on $\mathbf{R}^n$ is equivalent to the space $U = \{L\colon L = w_2(q, \dot q) + w_1(q)\}$ with respect to the hierarchy, where w_2, w_1 are 2-cocycle and 1-cocycle correspondingly on the Lie algebra $\mathbf{R}^n$. (The tangent vectors on $\mathbf{R}^n$ can be identified with points.) In the same way like in (5.3–5.5) we come to the statement that every weakly $\mathcal{G}$-invariant Lagrangian in this case has the form

$$L = F(\dot q^1, \ldots, \dot q^n) + \text{ full derivative } + B_{ik} q^i \dot q^k + E_i q^i.$$

It describes the interaction with constant "magnetic" and "electric" fields. (Compare with (1.5)). The corresponding Noether charges are $N_i(q, \dot q, t) = \partial L/\partial \dot q^i - B_{ik} q^k - E_i t$. The corresponding gradation of the space $\mathcal{V}_{0.0}$ is the following:

$$\mathcal{V}_{4.1} = \mathcal{V}_{3.1} = \mathcal{V}_{2.1}, \mathcal{V}_{1.1} = \mathcal{V}_{0.1} = \mathcal{V}_{4.1} \oplus \mathbf{R}^{\frac{n(n-1)}{2}}, \mathcal{V}_{s.0} = \mathcal{V}_{s.1} \oplus \mathbf{R}^n.$$

This case is famous in literature as "arising of constant magnetic filed as central extension of translations algebra [2]."

Example 3. $so(3)$ algebra.

In this example we consider the Lie algebra $so(3)$ which is the special case of semisimple algebra. Let $M = \mathbf{R}^3$ be 3-dimensional linear space with cartesian coordinates (x^1, x^2, x^3). We consider first the pairs $[so(3), \mathbf{R}^3]$ and $[so(3), S^2]$ where S^2 is the sphere $x^i x^i = 1$ in $\mathbf{R}^3$ and the action of $so(3)$ on $\mathbf{R}^3$ is generated by the standard action of the group $SO(3)$ on $\mathbf{R}^3$: if $\{e_1, e_2, e_3\}$ is a basis in $so(3)$ such that $[e_i, e_j] = \varepsilon_{ijk} e_k$ then $\Phi(e_i) = \tilde L_i = -\varepsilon_{ijk} x^j \partial/\partial x^k$. For the pair $[so(3), \mathbf{R}^3]$ the hierarchy diagram is trivial because $SO(3)$ is semisimple compact group. (See the end of the Section IV.)

Alternatively one can see it by the following explicit calculations: From commutation relations it is evident that $H^1(so(3)) = H^2(so(3)) = 0$. Hence $K_0 = K_1 = K_2 = K_4 = 0$. If α_i is a cocycle with values in functions on S^2 then $0 = \delta\alpha = \tilde{L}_i\alpha_k - \tilde{L}_k\alpha_i - \varepsilon_{ijk}\alpha_k$. Hence $\tilde{L}^2\alpha_k = \tilde{L}_k(\tilde{L}_i\alpha_i) = \tilde{L}_kF$ and $\alpha_k = \delta\tilde{F}$ is coboundary where $\tilde{F} = \sum_l \frac{F^l}{l(l+1)}$. ($F^l$ is defined by the expansion over the spherical harmonics of F. The term $F^0 = 0$ because it leads to cocycle in constants and $H^1(so(3)) = 0$.) Hence $K_3 = 0$ also.

The calculations and the result are the same for the diagram $[so(3), \mathbf{R}^3]$.

We come to the result that all weakly $so(3)$ invariant Lagrangians of classical mechanics on R^3 and on S^2 are exhausted by $so(3)$-invariant ones (up to a full derivative).

Now bearing in mind the construction (4.6) we modify little bit this example considering instead the sphere S^2 the domain in it, the sphere without North pole (punctured sphere) $S^2\backslash N]$ $(x^3 \neq 1)$. Thus we come from the pair $[so(3), S^2]$ to the pair $[so(3), S^2\backslash N]$. In the same way we come to the pair $[so(3), \mathbf{R}^3\backslash L_+]$, taking out the ray L_+ $(x^1 = 0, x^2 = 0, x^3 \geq 0)$ from $\mathbf{R}^3$.

The essential difference of these pairs from the previous ones is that they cannot be generated by the action of the corresponding Lie group.

We perform the calculations for the diagram $\mathcal{D}([so(3), S^2\backslash N])$.

It is evident that for this diagram $K_0 = K_1 = K_2 = K_4 = 0$, also. Now we show that for this diagram $K_3 = \mathbf{R}$ and this hierarchy is full.

The stability algebra for this pair is one dimensional, hence from (4.6–4.8) it follows that $K_3 = 0$ or $K_3 = \mathbf{R}$. It remains to prove that K_3 is not trivial.

To show it we consider the Lagrangian L which corresponds to the differential form $A = -(1 + cos\theta)d\varphi$ on the punctured sphere $S^2\backslash N$. (θ, φ are spherical coordinates.) The two-form $dA = F = sin\theta d\theta d\varphi$ corresponding to its motion equations is $so(3)$-invariant, hence this Lagrangian is weakly $so(3)$-invariant. On the other hand it cannot be reduced to $so(3)$-invariant by redefinition on a full derivative df because $so(3)$-invariant 1-form on the sphere is equal to zero. Hence, because all other spaces K_s are equal to zero. this Lagrangian belongs to the floor 2_+. We come to the result:

$$K_3 = H^1(so(3), \Lambda^0(S^2\backslash N)) = \mathbf{R} \quad \text{and } \phi_2(A) \in K_3 \neq 0, \tag{5.7}$$

For this special case the explicit realization of (4.6-4.8) is the following: We identify the vectors in $\mathbf{R}^3$ with the vectors in the linear space of the Lie algebra $so(3)$ by the linear map γ: $(x^1, x^2, x^3) \to x^1e_1 + x^2e_2 + x^3e_3$. For any point $x \in S^2$ the corresponding stability subalgebra is generated by $\gamma(x)$. To (4.6–4.7) corresponds the following statement: If α is a 1-cocycle with values in functions on the punctured sphere then

$$\alpha(x, \gamma(x)) = x^i\alpha_i(x) \text{ is a constant on the sphere,} \tag{5.8a}$$

$$\text{this constant is equal to zero iff this cocycle is a coboundary.} \tag{5.8b}$$

(This statement can be easily proved in a straightforward way without using (4.6,4.7)).

We proved that $K_3 = \mathbf{R}$ and all other K_s are equal to zero and presented in (5.7) the Lagrangian with non-trivial image in K_3. Hence the hierarchy diagram $\mathcal{D}([so(3), S^2\backslash N])$ is full on all the floors and the space of classiclal mechanics Lagrangians is equivalent to the one-dimensional space $U = \{L\colon L = -q(1 + cos\theta\dot{\varphi})\}$ with respect to this hierarchy. So using (4.3) we arrive at the statement that every weakly $so(3)$ invariant Lagrangian on the

punctured sphere has the form

$$L = L_{inv} + \text{full derivatives} - g(1 + cos\theta)\dot{\varphi}\,. \tag{5.9}$$

In the case $g \neq 0$ it belongs to the floor 2_+ of the hierarchy.

The calculations for the diagram $\mathcal{D}[so(3), \mathbf{R}^3 \backslash l_+]$ are analogous and the result is the same: every weakly $so(3)$ invariant Lagrangian on the $\mathbf{R}^3 \backslash l_+$ has the form (5.9).

One can see that in the case if L_{inv} is free particle Lagrangian, then (5.9) corresponds to the Lagrangian which describes the interaction of particle with Dirac monopole.

The explicit calculations for (5.7) give that $\phi_2(L)$ for the Lagrangian (5.9) is equal to the cohomology class in $H^1(so(3), S^2 \backslash N)$ of the following cocycle:

$$\alpha_1 = -gctg\frac{\theta}{2}cos\varphi,\ \alpha_2 = -gctg\frac{\theta}{2}sin\varphi,\ \alpha_3 = g, \quad (\delta L = d\alpha,\ \delta\alpha = 0) \tag{5.10}$$

and $\alpha_i x^i = -g$.

Finally we make the following remark about the Lagrangian (5.9)

Via stereographic projection of the punctured sphere on $\mathbf{R}^2$ one comes from the pair $[so(3], S^2 \backslash N]$ to the pair $[so(3), \mathbf{R}^2]$, where the fundamental vector field corresponding to e_3 corresponds to rotations and fundamental vector fields corresponding to e_1, e_2 correspond to non-linear infinitesimal transformations. The weakly $so(3)$-invariant Lagrangian (5.9) transforms to the

$$L = \frac{m(\dot{u}^2 + \dot{v}^2)}{2(1 + u^2 + v^2)^2} + g\frac{u\dot{v} - v\dot{u}}{1 + u^2 + v^2} \tag{5.11}$$

in the case if L_{inv} is free particle Lagrangian.

The Lagrangian (5.11) in the case $g = 0$ is strictly related with the Lagrangian which describes the interaction of free particle in 2-dimensional plane with Coulomb potential. To the vector fields $\tilde{e}_1, \tilde{e}_2$ correspond so called hiiden symmetries of Coulomb interaction which lead to Runge–Lentz vector [19]. So, Lagrangian (5.11) leads to the Lagrangian which possesses essentially generalized hidden symmetries of two-dimensional Coulomb potential. These consideration deal with so called higher symmetries which are not in the frame of this paper.

Example 4. Galilean and Poincaré Lie algebras

To threat these algebras simultaneously we consider 1-parametric family of Poincaré Lie algebras $\mathcal{G}(\mathcal{P}_c)$ (c is the "velocity of light"). Their action (2.1) on the space $\mathbf{R}^4$ with cartesian coordinates (t, x^1, x^2, x^3) is generated in a standard way via the following fundamental vector fields:

$$\tilde{p}_0 = \frac{\partial}{\partial t},\ \tilde{p}_i = \frac{\partial}{\partial x^i},\ \tilde{B}_i = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}x^i\frac{\partial}{\partial t},\ \tilde{L}_i = -\varepsilon_{ijk}x^j\frac{\partial}{\partial x^k}\,. \tag{5.12}$$

which correspond to its basis. The relations (5.12) define the pair $[\mathcal{G}(\mathcal{P}_c), \mathbf{R}^4]$.

In the case $c \to \infty$ Lie algebra $\mathcal{G}(\mathcal{P}_c)$ is contracted to the Lie algebra $\mathcal{G}(\Gamma)$ of Galilean group (non-relativistic limit) which we denote also by $\mathcal{G}(\mathcal{P}_\infty)$. (All the commutation

relations of basis vectors in $\mathcal{G}(\mathcal{P}_c)$ do not depend on c, except the relations $[\tilde{B}_i, \tilde{B}_k] = -1/c^2 \varepsilon_{ijk} \tilde{L}_k, [\tilde{p}_i, \tilde{B}_k] = -1/c^2 p_0 \delta_{ik}$ which tend to zero if c tends to zero.)

Correspondingly to (5.12) the action of the Galilean Lie algebra $\mathcal{G}(\mathcal{P}_\infty)$ on $\mathbf{R}^4$ is generated via the vector fields:

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x^i}, \quad t\frac{\partial}{\partial x^i}, \quad L_i = -\varepsilon_{ijk} x^j \frac{\partial}{\partial x^k} \,. \tag{5.13}$$

It defines the pair $[\mathcal{G}(\mathcal{P}_\infty), \mathbf{R}^4]$. (The vector field corresponding to Lorentz boost transforms to vector field corresponding to special Galilean transformation.)

The first two cohomologies groups for algebras $\mathcal{G}(\mathcal{P}_c)$ are

$$\begin{array}{ll} H^1(\mathcal{G}(\mathcal{P}_c)) = 0, \quad H^2(\mathcal{G}(\mathcal{P}_c)) = 0, & \text{if } c \neq \infty \\ H^1(\mathcal{G}(\mathcal{P}_\infty)) = 0, \quad H^2(\mathcal{G}(\mathcal{P}_\infty)) = \mathbf{R}. & \end{array} \tag{5.14}$$

The second cohomology group of the Galilean Lie algebra is generated by 2-cocycle c_B (Bargmann cocycle) whose non-vanishing components in the basis (5.13) are only

$$c_B(p_i, B_j) = -c_B(B_j, p_i) = \delta_{ij}. \tag{5.15}$$

The relations (5.14) make trivial the calculations of the all spaces K_s except the space K_3 for the hierarchy diagram $\mathcal{D}(\mathcal{G}(\mathcal{P}_c), \mathbf{R}^4)$. From the formula (4.6) it follows that $K_3 = 0$ because the stability subalgebra of every point (t_0, x_0^i) in $\mathbf{R}^4$ is isomorphic to the subalgebra generated by the vectors (L_i, B_j) which has only trivial 1-cocycles.

Hence for the hierarchy diagram $\mathcal{D}(\mathcal{G}(\mathcal{P}_c), \mathbf{R}^4)$ all the spaces K_s are equal to zero, except K_2 which is equal to $\mathbf{R}$ for Galilean algebra and which is equal to zero for Poincaré algebra.

We see that the hierarchy diagram $\mathcal{D}(\mathcal{G}(\mathcal{P}_c), \mathbf{R}^4)$ for Poincaré algebra is trivial. For Galilean algebra in the diagram $\mathcal{D}(\mathcal{G}(\mathcal{P}_\infty), \mathbf{R}^4)$ only the floors $1_+, 4_+$ can be non-empty.

It has to be noted that the space of Lagrangians $L(t, x^i, dt/d\tau, dx^i/d\tau)$ in $\mathbf{R}^4$ is more wide that the space of classical mechanics Lagrangians $L(x^i, dx^i/dt)$ on the configuration space $\mathbf{R}^3$. To every Lagrangian in $\mathbf{R}^3$ according to (2.16) corresponds Lagrangian which is a density in $\mathbf{R}^4$. On the other hand to every Lagrangian–density L in $\mathbf{R}^4$ which does not depend explicitly on time corresponds the classical mechanics Lagrangian, if we put the parameter $\tau = t$. For example to the Lagrangian of free non-relativistic particle corresponds the density in $\mathbf{R}^4$ $L_{nonrel} = m\dot{x}^i\dot{x}^i/2\dot{t}$ and to the Lagrangian of free relativistic particle corresponds the density $L_{rel}(c) = -mc\sqrt{c^2\dot{t}^2 - \dot{x}^i\dot{x}^i}$. (The $\dot{x}^i, \dot{t}$ means derivatives of x, t with respect to the parameter τ.) The Lagrangian $L_{rel}(c) + mc^2\dot{t}$ which differs from $L_{rel}(c)$ on the full derivative tends to L_{nonrel} if $c \to \infty$.

The Lagrangian $L_{rel}(c)$ is $\mathcal{G}(\mathcal{P}_c)$-invariant. $L_{rel}(c)$ is the unique (up to multiplier) $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangian in the space of densities on $\mathbf{R}^4$.

On the other hand it is evident that there are no $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangians in the space of densities on $\mathbf{R}^4$, except trivial ones. The Lagrangian L_{nonrel} is weakly $\mathcal{G}(\mathcal{P}_\infty)$-invariant:

$$\mathcal{L}_{p_0} L_{nonrel} = \mathcal{L}_{p_i} L_{nonrel} = \mathcal{L}_{L_i} L_{nonrel} = 0, \, \mathcal{L}_{B_i} L_{nonrel} = m\dot{x}^i \,. \tag{5.16}$$

From (5.16) it follows that $\delta L = d\alpha$ where the values of the cocycle α on the basis vectors B_i are equal to mx^i, on all other basis vectors α is equal to zero. Hence

$$\delta\alpha = mc_B, \tag{5.17}$$

where c_B is Bargmann cocycle (5.15).

We proved before that for the diagram $\mathcal{D}([\mathcal{G}(\mathcal{P}_\infty)], \mathbf{R}^4)$, $K_2 = \mathbf{R}$ and all other K_s are equal to zero. On the other hand it follows from (5.16,5.17) that the homomorphism ϕ_2 for the diagram $\mathcal{D}([\mathcal{G}(\mathcal{P}_\infty)], \mathbf{R}^4)$ has non-trivial image in K_2 at the Lagrangian L_{nonrel}. Hence the space of all weakly $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangians on $\mathbf{R}^4$ is equivalent to the one-dimensional space generated by the Lagrangian of free non-relativistic particle.

We come to the following conclusion:

Every weakly $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangian which is a density in $\mathbf{R}^4$ belongs to the floor 1_+ and is proportional to L_{nonrel} (up to full derivative and a constant). The floor 4_+ contains only trivial Lagrangians.

Correspondingly for the Poincaré algebra every weakly $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangian-density coincides (up to full derivatives) with $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangian $L_{rel}(c)$.

On one hand to the contraction of Poincaré algebra to Galilean algebra corresponds the arising of Bargmann cocycle. On the other hand the unique non-trivial component $\mathcal{V}_{4.1}$ of the hierarchy diagram for Poincaré algebra transforms to the unique non-trivial component $\mathcal{V}_{1.1}$ of the hierarchy diagram for Galilean algebra.

The vanishing of $H^2(\mathcal{G}(\mathcal{P}_c))$ is the reason why in relativistic quantum mechanics the projective representation of Poincaré symmetries in the space of states (which are rays in linear Hilbert space) can be reduced to linear one and because of (5.15) it is not the case in non-relativistic mechanics. The considerations of this example are reflection of this phenomenon.

VI Discussions

The problem considered here and the technique which we used to study it can be generalized in a few directions. The considerations of this paper can be easily translated to Hamiltonian language. One can consider the classification of Lagrangians not only for symmetries induced by point transformations of configuration space but by the so called higher symmetries. For example from this point of view it is interesting to analyze the generalized Runge–Lentz symmetries (see the end of Example 3 in Section V).

It is interesting to apply this method to supersymmetrical case [12]. It seems to be interesting also to analyze the phenomena of spin-like transformations (1.9) arising for Lagrangians from the second floor of the hierarchy (4.2), in order to apply it to Dirac monopoles [20].

We hope that a generalization of this method on field theory Lagrangians will be fruitful. From this point of view we want to note the relations of our considerations with the problem of the Ward identities anomaly absence for field theory Lagrangians which possess classically the given symmetry [21,10].

To develop this technique for field theory Lagrangians, the first order formalism and multisymplectic formalism become very useful [19]. We would wish to develop these considerations on the firm ground of investigations of A.M. Vinogradov and his collaborators [22].

On the other hand, in our opinion, the method considered in this paper maybe is more important than the problem we applied it to.

We give here only three examples, one of them pure mathematical, where the calculations of double comlex cohomology (the method we use in this paper) make a bridge between the corresponding structures.

1. *The calculation of de Rham cohomology in terms of Chech cohomology.*

When manifold M is covered by the family $\{U_\alpha\}$ of open sets one can consider Chech cohomology of this covering. Then one can consider double complex of q-forms which are defined on the sets $\{U_\alpha\}$. The differential Q of this complex is the sum of de Rham exterior differential and Chech differential. Considering the differential Q "perturbatively" around Chech differential one arrives naturally at de Rham cohomology of M, hence the "perturbative" calculations around de Rham differential lead in general to calculation of spectral sequence which tends to de Rham cohomology of M. In the case if the covering is Leray covering, i.e. all the sets and their intersections are convex connected sets then Chech cohomology coincide with de Rham one; the application of Poincaré lemma reduces spectral sequence calculations to trivial resolutions of descent equations. But practically it is more convenient to use for calculations a suitable covering which generally is not a Leray covering. (See for details e.g. [23].).

2. *The relations between Hamiltonian reduction method and BRST cohomology for classical mechanics*

One can say that the relations between these two methods are encoded in the cohomology of double complex differential $Q = \partial + \delta$ in the case if constraints form Lie algebra (so called closed groups.) Here ∂ corresponds to Koszul differential of complex generated by constraints and δ is differential corresponding to Hamiltonian vector fields which are induced by these constraints. Perturbative expansion of Q around δ leads to standard Hamiltonian methods, and expansion around ∂ leads to BRST. In the case if constraints form so called open group, one has to consider the corresponding filtered space instead of this double complex [3,4,6]. This approach seems to be very fruitful.

3. *Local BRST Cohomology*

Considering BRST physical observables as integrals of local functions one comes naturally to differential $Q = s + d$, where s is BRST differential, acting on integrand which is local function and d is the usual de Rham differential. It turns out that the consideration of cohomology of this double complex is a very powerful tool for BRST cohomology investigations in field theory, especially in Lagrangian framework. (See [8,9,10,24] and citations there).

In spite of these examples one has to note that the method of spectral sequences was not used actively in these calculations.

May be at first the method of spectral sequences was applied in physics by J. Dixon in [8] in analysis of local BRST cohomology. In series of works the so called method of descent equations which is in fact a special case reminiscent of this technique was applied successfully to these problems. (See the review [10] and the papers citated there). Nowadays the technique of spectral sequences seems to be not very popular in theoretical physics. We hope to pay attention to importance of this technique which is used here in a simple physical frame. In principal using the method "Deus ex machina" one can formulate the hierarchy without using explicitly the method used in this paper which indeed seems to be very tedious. But in our opinion this method is inherent to this problem and it is

the adequate technique in other important problems such as constrained dynamics theory; it may have useful applications in future.

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Appendix 1. Lie algebra cohomologies

Let $\mathcal{G}$ be Lie algebra and A be a linear space which is module on $\mathcal{G}$, i.e. the action of $\mathcal{G}$ on A which respects the structure of the Lie algebra $\mathcal{G}$ and the space A is defined:

$$
\begin{aligned}
&h \in \mathcal{G}, m \in A \quad (h,m) \to h \circ m \in A: \\
&\qquad (\lambda h_1 + \mu h_2) \circ m = \lambda (h_1 \circ m) + \mu (h_2 \circ m), \quad (\lambda, \mu \in \mathbf{R}) \\
&\qquad h \circ (\lambda m_1 + \mu m_2) = \lambda (h \circ m_1) + \mu (h \circ m_2), \\
&h_1 \circ (h_2 \circ m) - h_2 \circ (h_1 \circ m) = [h_1, h_2] \circ m\,.
\end{aligned}
\tag{A1.1}
$$

([,] defines commutator in $\mathcal{G}$. A and $\mathcal{G}$ are linear spaces on $\mathbf{R}$).

The complex $(C^q(\mathcal{G}, A), \delta)$ of cochains can be defined in the following way. Let $C^q(\mathcal{G}, A)$ be a space of skewsymmetric q-linear functions on $\mathcal{G}$ (q-cochains) which take values in A (If $q = 0$, $C^0(\mathcal{G}, A) = A$). $\mathcal{G}$-differential δ on $\{C^q\}$ $\delta\colon C^q \to C^{q+1}$, $\delta^2 = 0$ is defined in the following way:

$$
\begin{aligned}
&\delta\colon C^0 \to C^1 \quad (\delta c)(h) = h \circ c, (c \in C^0 = A) \\
&\delta\colon C^1 \to C^2 \quad (\delta c)(h_1, h_2) = h_1 \circ c(h_2) - h_2 \circ c(h_1) - c([h_1, h_2]),
\end{aligned}
\tag{A1.2}
$$

and so on:

$$
\begin{aligned}
\delta\colon C^q \to C^{q+1} \quad (\delta c)(h_1, \ldots, h_{q+1}) = &\sum_{1 \le i \le q+1} (-1)^{i+1} h_i \circ c(h_1, \ldots, \hat{h}_i \ldots, h_{q+1}) + \\
&\sum_{1 \le i < j \le q+1} (-1)^{i+j} c([h_i, h_j], h_1, \ldots, \hat{h}_i, \ldots, \hat{h}_j \ldots, h_{q+1})
\end{aligned}
$$

($\hat{h}_i$ means omitting of the variable h_i)). The cohomologies $H^q(\mathcal{G}, A)$ of the complex $(\{C^q\}, \delta)$ are called cohomologies of Lie algebra $\mathcal{G}$ with coefficients in the module A. (See in details for example [23].)

$$
H^q(\mathcal{G}, A) = \left(\mathbf{ker}\,\delta\colon C^q \to C^{q+1}\right) / \left(\mathbf{Im}\,\delta\colon C^{q-1} \to C^q\right)\,.
$$

If module A is $\mathbf{R}$ and $\mathcal{G}$ acts trivially on it: $h \circ \lambda = 0$, $C^q(\mathcal{G}, \mathbf{R})$ is denoted by $C^q(\mathcal{G})$ and correspondingly $H^q(\mathcal{G}, \mathbf{R})$ is denoted by $H^q(\mathcal{G})$. In this case cochains are constant antisymmetrical tensors and $\mathcal{G}$-differential δ is expressed only via structure constants $\{t^n_{ik}\}$ of Lie algebra $\mathcal{G}$.

$H^0(\mathcal{G}) = \mathbf{R}$, $H^1(\mathcal{G})$ is defined by the solutions of the equation $c^m_{ik} b_m = 0$ and it is nothing but the space dual to the $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$.

In a case if $\mathcal{G}$ is abelian $H^q(\mathcal{G}) = C^q(\mathcal{G}) = (\wedge \mathcal{G}^*)^q$ where $\mathcal{G}^*$ is the linear space dual to the linear space of $\mathcal{G}$.

In a case if $\mathcal{G}$ is semisimple Lie algebra then $H^1\mathcal{G}$=$H^2\mathcal{G}$=0. This statement is valid in a general case too. Very important Whitehead lemmas state that if $\mathcal{G}$ is semisimple Lie algebra then $H^1(\mathcal{G}, A) = H^2(\mathcal{G}, A) = 0$ in the case if A is an arbitrary module which is *finite-dimensional* vector space on $\mathbf{R}$ [23]

Appendix 2. Double complex and its spectral sequences.

Now we give a brief sketch on the topic how to apply spectral sequences technique for calculations of cohomology of double complexes. (See for the details for example [23].)

Let $E^{**} = \{E^{p \cdot q}\}$ $(p, q = 0, 1, 2, ...)$ be a family of abelian groups (modules, vector spaces) on which are defined two differentials ∂_1 and ∂_2 which define complexes in rows and in columns of $E^{*\cdot *}$ and which commute with each other:

$$\partial_1 : E^{p \cdot q} \to E^{p \cdot q+1}, \partial_1^2 = 0,\ \partial_2 : E^{p \cdot q} \to E^{p+1 \cdot q}, \partial_2^2 = 0,\ \partial_1 \partial_2 = \partial_2 \partial_1 \,. \qquad (A2.1)$$

$\{E^{**}, \partial_1, \partial_2\}$ is called double complex.

(It is convenient to consider $E^{p \cdot q}$ for all integers p and q fixing that $E^{p \cdot q} = 0$ if $p < 0$ or $q < 0$.)

One can consider "antidiagonals": $\mathcal{D}^m = \{E^{p \cdot m-p}\}$ $(p = 0, 1, ..., m)$ which form complex with differential

$$Q = (-1)^q \partial_2 + \partial_1 \qquad (A2.2)$$

which evidently obeys to condition $Q^2 = 0$.

$$0 \to \mathcal{D}^0 \xrightarrow{Q} \mathcal{D}^1 \xrightarrow{Q} \mathcal{D}^2 \to \dots . \qquad (A2.3)$$

The cohomologies $H^m(Q)$ of this complex are called the cohomologies of double complex $(E^{**}, \partial_1, \partial_2)$.

The rows and the columns complexes define the cohomologies $H(\partial_1)$ and $H(\partial_2)$ of E^{**}.

One can consider the filtration corresponding to the double complex $\{E^{*\cdot *}, \partial_1, \partial_2\}$

$$\ldots \subseteq X^m \subseteq X^{m+1} \subseteq \ldots \subseteq X^1 \subseteq X^0 \qquad (A2.4)$$

$$\text{where} \qquad X^k = \bigoplus_{q \geq 0, p \geq k} E^{p \cdot q} \qquad (A2.5)$$

and sequence of the spaces $\{E_r^{p\cdot q}\}$ $(r = 0, 1, 2, \ldots$ corresponding to this filtration

$$E_r^{p\cdot q} = Z_r^{p\cdot q} \big/ B_r^{p\cdot q} \quad (E_0^{p\cdot q} = E^{p\cdot q})\,. \tag{A2.6}$$

In (A2.6) $Z_r^{p\cdot q}$ ("r-th order cocycles") is the space of the elements in $E^{p\cdot q}$ which are leader terms of cocycles of the differential Q up to r–th order w.r.t. the filtration (A2.4), i.e.

$$\{Z_r^{p\cdot q}\} = \{E_r^{p\cdot q} \ni c\colon \quad \exists \tilde{c} = c(mod\, X_{p+1})\,\text{such that}\, Q\tilde{c} = 0(mod\, X_{p+r})\}\,. \tag{A2.7}$$

It means that there exists $\tilde{c} = (c, c_1, c_2, \ldots, c_{r-1})$ where $c_i \in E^{p+i\cdot q-i}$ such that $Q(c, c_1, c_2, \ldots, c_{r-1}) \subseteq X_{p+r}$:

$$\partial_1 c = 0, \partial_2 c = \partial_1 c_1, \partial_2 c_1 = \partial_1 c_2, \ldots, \partial_2 c_{r-2} = \partial_1 c_{r-1}, \,\text{so}\, Q\tilde{c} = \partial_2 c_{r-1} \in X_{p+r}\,.$$

Correspondingly $B_r^{p\cdot q}$ is the space of up to r–th order borders:

$$\{B_r^{p\cdot q}\} = \{E_r^{p\cdot q} \ni c\colon \quad \exists \tilde{b} \in X_{p-r+1}\,\text{such} \quad \text{that}\, Q\tilde{b} = c\,. \tag{A2.8}$$

It means that there exist $\tilde{c} = (b_0, b_1, b_2, \ldots, b_{r-1})$ where $b_i \in E^{p-i\cdot q+i}$ and $Q(b_0, b_1, b_2, \ldots, b_{r-1}) = c$:

$$\partial_1 b_0 + \partial_2 b_1 = c, \partial_1 b_1 + \partial_2 b_2 = 0, \partial_1 b_2 + \partial_2 b_3 = 0, \ldots, \partial_1 b_{r-1} = 0\,. \tag{A2.9}$$

For example $E_1^{p\cdot q} = H(\partial_1, E^{p\cdot q})$.

We denote by $[c]_r$ the equivalence class of the element c in the $E_r^{p\cdot q}$ if $c \in Z_r^{p\cdot q}$.

It is easy to see that the sequence $\{E_r^{p\cdot q}\}$ $r = 0, 1, 2, \ldots$ is stabilized after finite number of the steps: $(E_{r_0}^{p\cdot q} = E_{r_0+1}^{p\cdot q} = \ldots = E_\infty^{p\cdot q}$, where $r_0 = max\{p+1, q+1\}$.

Let $H^m(Q, X_p)$ be cohomologies groups of double complex truncated by filtration (A2.4) (we come to $H^m(Q, X_p)$ considering $\{\mathcal{D} \cap X^p, Q\}$ as subcomplex of (A2.3), $H^m(Q) = H^m(Q, X^0)$. We denote by ${}_{(p)}H^m(Q)$ the image of $H^m(Q, X_p)$ in $H(Q)$ under the homomorphism induced by the embedding $\mathcal{D} \cup X_p \to \mathcal{D}$. The spaces ${}_{(p)}H^m(Q)$ are embedded in each other

$$0 \subseteq {}_{(m)}H^m(Q) \subseteq {}_{(m-1)}H^m(Q) \subseteq \ldots {}_{(1)}H^m(Q) \subseteq {}_{(0)}H^m(Q) = H^m(Q)\,. \tag{A2.10}$$

The spaces $E_\infty^{p\cdot q\cdot}$ considered above are related with (A2.10) by the following relations:

$$E_\infty^{p.m-p} =_{(p)} H^m(Q) \big/ {}_{(p+1)}H^m(Q)\,. \tag{A2.11}$$

In particular $E_\infty^{0.m}$ is canonically embedded in $H^m(Q)$.

The formula (A2.11) is the basic formula which expresses the cohomology $H(Q)$ of the double complex $\{E^{p\cdot q}, \partial_1, \partial_2\}$ in terms of $\{E_\infty^{p\cdot q}\}$. From (A2.10, A2.11) it follows that

$$H^m(Q) \simeq \bigoplus_{i=0}^{m} E^{p-i.i}\,. \tag{A2.12}$$

The essential difference of (A2.12) from (A2.11) is that in (A2.12) the isomorphism of l.h.s. and of r.h.s. *is not canonical*.

The importance of the sequence $\{E_r^{*.*}\}$ $(r = 0, 1, 2, \ldots)$ is explained by the fact that its terms (and so $\{E_\infty^{*.*}\}$) can be calculated in a recurrent way. Namely one can consider differentials (See for details [23.]) d_r: $E_r^{p.q} \to E_r^{p+r.q+1-r}$ such that $\{E_r^{*.*}, d_r\}$ form spectral sequence, i.e.

$$E_{r+1}^{*.*} = H(d_r, E_r^{*.*}). \qquad (A2.13)$$

The differentials d_r are constructed in the following way: $d_0 = \partial_1$: $E^{p.q} = E_0^{p.q} \to E^{p.q+1} = E_0^{p.q+1}$.
If $c \in E^{p.q}$ and $\partial_1 c = 0 \leftrightarrow [c]_1 \in E_1^{p.q}$ then $d_1[c] = [\partial_2 c]$, d_1: $E_1^{p.q} \to E_1^{p+1.q}$.
In general case for $[c]_r \in E_r^{p.q}$ $d_r[c]_r = [Q\tilde{c}]_r$ d_r: $E_r^{p.q} \to E_1^{p+r.q+1-r}$,

where $\tilde{c}$: $c - \tilde{c} \in X^{p+r}$ (see the definition (A2.7) of $Z_r^{p.q}$).

One can show that definition of d_r is correct, $d_r^2 = 0$ and (A2.13) is obeyed [23].

Using (A2.13) one come after finite number of steps to $E_\infty^{p.q}$ calculating each $E_r^{p.q}$ as the cohomology group of the $E_{r-1}^{p.q}$: $E_1^{p.q} = H(d_0, E^{p.q})$, $E_2^{p.q} = H(d_1, E_1^{p.q}$ and so on.

The spaces $E_r^{p.q}$ can be considered intuitively as r–th order (with respect to differential ∂_2) cohomologies of differential Q . The operator ∂_1 is zeroth order approximation for differential Q. The calculations of $E_\infty^{p.q}$ via (A2.13) can be considered as perturbational calculations.

One can develop this scheme considering in perturbative calculations not the operator ∂_1, but ∂_2 as zeroth order approximation.

Instead filtration (A2.4) one has consider the "transposed" filtration

$$\ldots \subseteq {}^tX^m \subseteq {}^tX^{m+1} \subseteq \ldots \subseteq {}^tX^1 \subseteq X^0 \qquad (A2.14)$$

$$\text{where} \qquad {}^tX^k = \bigoplus_{p\geq 0, q\geq k} E^{p.q}$$

and corresponding transposed spaces $\{{}^tE_r^{p.q}\}$. For example

$$E_1^{p.q} = H(\partial_1, E^{p.q}), \quad {}^tE_r^{p.q} = H(\partial_2, E^{p.q})\,.$$

Instead spectral sequence $\{E_r^{*.*}, d_r\}$ one has to consider transposed spectral sequence $\{{}^tE_r^{*.*}, {}^td_r\}$:

$$d_0 = \partial_1, \to {}^td_0 = \partial_2\,; d_1[c]_1 = [\partial_2 c]_1, \to {}^td_1[c]_1 = [\partial_1 c]_1\,,$$

and so on.

The relations between spaces $\{E_\infty^{p.q}\}$ and $\{{}^tE_\infty^{p.q}\}$ which express in different ways the cohomology $H(Q)$ is one of the applications of the method described here.

Example. Let $\mathbf{c} = (c_0, c_1.c_2)$ where $c_0 \in E^{0.2}, c_1 \in E^{1.1}, c_2 \in E^{2.0}$ be cocycle of the differential Q: $Q(c_0, c_1.c_2) = 0$ i.e. $\partial_1 c_0 = 0, \partial_2 c_0 = -\partial_1 c_1, \partial_2 c_1 = \partial_1 c_2$. To the leading term c_0 of this cocycle w.r.t. the filtration (A2.4) corresponds the element $[c_0]_\infty$ in $E_\infty^{0.2}$ which represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{0.2}$.

In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (0, c_1', c_2')$ has a solution, i.e. the leading term c_0 of the cocycle $\mathbf{c}$ can be cancelled by changing of this cocycle on a coboundary, then the element $[c_1']_\infty \in E_\infty^{1.1}$ represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{1.1}$.

In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (0, 0, \tilde{c}_2)$ have a solution, i.e. the leading term and next one both can be cancelled, by redefinition on a coboundary, then $[\tilde{c}_2]_\infty \in E_\infty^{2.0}$ represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{2.0}$.

To put correspondences between the cohomology class of the cocycle $\mathbf{c}$ and corresponding elements from transposed spaces ${}^tE_\infty^{0.2}$, ${}^tE_\infty^{1.1}\,{}^tE_\infty^{1.1}$ we have to do the same, changing only the definition of leading terms, which we have to consider now w.r.t. the filtration (A2.14).

To the leading term c_2 of this cocycle w.r.t. the filtration (A2.14) corresponds the element $[c_2]_\infty$ in ${}^tE_\infty^{2.0}$ which represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE_\infty^{2.0}$. In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (c_0', c_1', 0)$ has a solution, i.e. the leading term c_0 of the cocycle $\mathbf{c}$ can be cancelled by changing of on a coboundary, then the element $[c_1']_\infty$ represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE_\infty^{1.1}$. In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (\tilde{c}_0, 0, 0)$ has a solution, then $[\tilde{c}_0]$ represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE_\infty^{0.2}$.

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